UNIVERSITATEA DE VEST DIN TIMIŞOARA
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TEZĂ DE DOCTORAT

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Existence and Multiplicity Results for Discrete $p(\cdot)$-Laplacian and for Singular ϕ-Laplacian

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Contents

Introduction

This thesis is concerned with existence and multiplicity of solutions for boundary value problems involving the discrete $p(\cdot)$-Laplacian operator and the singular ϕ-Laplacian operator. Our approach mainly relies on the direct method in calculus of variations, Saddle Point Theorem, Mountain Pass Theorem, some abstract three critical points results, monotone iterative techniques, the method of lower and upper solutions, as well on Leray-Schauder degree type arguments.

The work is divided in three chapters. The first one consists of four paragraphs, while the second and the third are each structured in three sections.

Some preliminary notions and results which are needed throughout the thesis are introduced in *Chapter 1*. In the first paragraph of the chapter some basic properties of lower semi-continuous, coercive and convex functions are recalled. Then, in the following two paragraphs, we make a short overview on some results related to the classical and Szulkin's critical point theory. In the final paragraph we provide two abstract three critical points theorems.

In *Chapter 2* we use the critical point theory to establish the existence and multiplicity of solutions for discrete $p(\cdot)$-Laplacian equations subjected to periodic, Neumann and general potential type boundary conditions. As usually, the idea is to transfer the problem of the existence of solutions into a problem of finding critical points for the corresponding Euler-Lagrange functional. So, for given $s \in (1, \infty)$, h_s will be the homeomorphism defined by $h_s(x) = |x|^{s-2}x$, for all $x \in \mathbb{R}$. If $a, b \in \mathbb{N}$ with $a < b$, then $\mathbb{Z}[a, b]$ will denote the discrete interval $\{a, a+1, \dots, b\}$. Also, given T a positive integer and a function $p : \mathbb{Z}[0, T] \to (1, \infty)$, $\Delta_{p(\cdot)}$ will stand for the discrete $p(\cdot)$-Laplacian operator; recall, that is

$$\begin{aligned}\Delta_{p(k-1)}x(k-1) :&= \Delta(h_{p(k-1)}(\Delta x(k-1))) \\ &= h_{p(k)}(\Delta x(k)) - h_{p(k-1)}(\Delta x(k-1)),\end{aligned}$$

where $\Delta x(k) = x(k+1) - x(k)$ is the forward difference operator.

Various problems in applied mathematics lead to consideration of difference equations (see e.g., R.P. Agarwal [1], W.G. Kelly and A.C. Peterson [89] and the references therein). Recently, much attention has been paid to their study. Certainly this is largely due to the dynamic development of computer technology. For this reason, the research concerning boundary value problems involving the discrete $p(\cdot)$-Laplacian operator has gained an increasing popularity in very recent time. According to most of the papers in this field, it seems that the study was initiated in 2009 by M. Mihăilescu *et al.* in [110], where some eigenvalue problems for discrete $p(\cdot)$-Laplacian were investigated. After that, various existence and multiplicity results for related discrete anisotropic boundary value problems were obtained. Most of them concern with homogeneous Dirichlet problems. In this direction we refer the reader to [66]-[69], [90], [100]. But, the study of discrete $p(\cdot)$-Laplacian equations subjected to other boundary conditions seems lagging behind. To the best of our knowledge, only A. Guiro *et al.* [72] and B. Koné and S. Ouaro [91] proved existence of solutions for some discrete anisotropic Neumann and mixed boundary value problems. Hence, the present chapter aims to partly fill the gap in this area.

Concerning continuous anisotropic problems, it is worth to point out that these were intensively studied in the last decade by many authors. We refer the reader e.g. to [51], [53], [56]-[62], [64], [65], [74], [85], [86], [95], [107]-[109], [127]-[130] for existence and multiplicity results of the $p(x)$-Laplacian operator, using the theory of variable exponent Lebesque and Sobolev spaces – see X.L. Fan and D. Zhao [63], O. Kováčik and J. Rákosník [92] and the recent monograph of L. Diening *et al.* [52].

In Section 2.1 we deal with difference equations of type

$$-\Delta_{p(k-1)}x(k-1) = f(k, x(k)), \quad (\forall)\ k \in \mathbb{Z}[1, T], \tag{1}$$

subjected to the general potential boundary condition

$$(h_{p(0)}(\Delta x(0)), -h_{p(T)}(\Delta x(T))) \in \partial j(x(0), x(T+1)), \tag{2}$$

where $f : \mathbb{Z}[1, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $j : \mathbb{R} \times \mathbb{R} \to (-\infty, +\infty]$ is convex, proper, lower semi-continuous and ∂j denotes the subdifferential of j. First, we emphasize that the potential boundary condition (2) recovers the classical ones and hence a unified approach of all the classical boundary conditions is provided (see Remark 2.1.15 for details).

Szulkin's critical point theory [121] is employed when dealing with problem (1), (2). So, the associated Euler-Lagrange functional is defined on the finite dimensional space

$$X_{PO} := \{x : \mathbb{Z}[0, T+1] \to \mathbb{R}\}$$

(here, the index "PO" comes from "potential") by

$$\mathcal{I}(x) = \psi(x) - \sum_{k=1}^{T} F(k, x(k)), \quad (\forall)\ x \in X_{PO},$$

where $\psi : X_{PO} \to (-\infty, +\infty]$ is given by

$$\psi(x) = \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta x(k-1)|^{p(k-1)} + j(x(0), x(T+1)), \quad (\forall)\ x \in X_{PO}$$

and $F : \mathbb{Z}[1, T] \times \mathbb{R} \to \mathbb{R}$ is the primitive of f with respect to the second variable, i.e.,

$$F(k, t) = \int_0^t f(k, \tau) d\tau, \quad (\forall)\ k \in \mathbb{Z}[1, T],\ t \in \mathbb{R}. \tag{3}$$

First of all, we show in Proposition 2.1.4 that if $x \in X_{PO}$ is a critical point of the functional $\mathcal{I}$ in the sense of Szulkin, then x is a solution of problem (1), (2).

There are two main results in this section, namely Theorem 2.1.5 and Theorem 2.1.12. In the first one we use a minimization argument in order to prove the existence of at least one solution. Then, we obtain in Theorem 2.1.12 the existence of at least one nontrivial solution for problem

$$\begin{cases} -\Delta_{p(k-1)} x(k-1) + r(k) h_{p(k)}(x(k)) = f(k, x(k)), \quad (\forall)\ k \in \mathbb{Z}[1, T], \\ \\ (h_{p(0)}(\Delta x(0)), -h_{p(T)}(\Delta x(T))) \in \partial j(x(0), x(T+1)), \end{cases}$$

where $r : \mathbb{Z}[1, T] \to [0, \infty)$ is a given function. The main tool used here will be the Mountain Pass Theorem. The results from this section are obtained in [16]. Also, it is worth to point out that Theorem 3.1 and Theorem 4.4 in [122] – obtained for $p =$ constant, are immediate consequences of the above mentioned Theorem 2.1.5, respectively Theorem 2.1.12.

In Section 2.2 we are concerned with the existence of solutions for the periodic problem

$$\begin{cases} -\Delta_{p(k-1)} x(k-1) = f(k, x(k)), \quad (\forall)\ k \in \mathbb{Z}[1, T], \\ \\ x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T) \end{cases} \tag{4}$$

and for the Neumann problem

$$\begin{cases} -\Delta_{p(k-1)}x(k-1) = f(k,x(k)), \quad (\forall)\ k \in \mathbb{Z}[1,T], \\ \\ \Delta x(0) = 0 = \Delta x(T), \end{cases} \tag{5}$$

where as above $f : \mathbb{Z}[1,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function whose primitive F is defined in (3). To establish the existence results for problems (4) and (5) we shall use the classical critical point theory of Ambrosetti-Rabinowitz [5]. The solutions which we obtain appear either as minimizers or as saddle points of the corresponding energy functional

$$\mathcal{E}_X(x) = \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta x(k-1)|^{p(k-1)} - \sum_{k=1}^{T} F(k,x(k)), \quad (\forall)\ x \in X,$$

where

$$X = X_P := \{x : \mathbb{Z}[0,T+1] \to \mathbb{R} \mid x(0) = x(T+1)\},$$

when we refer to the periodic problem (4), respectively

$$X = X_N := \{x : \mathbb{Z}[0,T+1] \to \mathbb{R}\} \ (= X_{PO}),$$

in the case of the Neumann problem (5).

In Proposition 2.2.2 (resp. Proposition 2.2.4) we show that a function $x \in X_P$ (resp. $x \in X_N$) is a solution of problem (4) (resp. (5)) if and only if it is a critical point of $\mathcal{E}_{X_P}$ (resp. $\mathcal{E}_{X_N}$).

First, we deal with the case when no convexity assumptions are made on the nonlinearity f. So, if f is bounded and one of the following Ahmad-Lazer-Paul [4] type conditions:

$$\sum_{k=1}^{T} F(k,t) \to -\infty, \qquad \text{as } |t| \to \infty \tag{6}$$

or

$$\sum_{k=1}^{T} F(k,t) \to +\infty, \qquad \text{as } |t| \to \infty \tag{7}$$

holds true, we obtain in Theorem 2.2.9 (resp. Theorem 2.2.14) that problem (4) (resp. (5)) has at least one solution in X_P (resp. in X_N). Thus, if (6) is fulfilled, we shall prove that $\mathcal{E}_{X_P}$ (resp. $\mathcal{E}_{X_N}$) is coercive and then, by the direct method in the calculus of variations, problem (4) (resp. (5)) has at least one solution which minimizes $\mathcal{E}_{X_P}$ (resp. $\mathcal{E}_{X_N}$) on X_P (resp. on X_N). In the second case – when (7) is satisfied, we shall apply the Saddle Point Theorem. The situation of periodic potential F is discussed in Theorem

2.2.7 and Theorem 2.2.13.

A comparison result is obtained for a slightly more general version of problem (4):

$$\begin{cases} -\Delta_{p(k-1)}x(k-1) = f(k,x(k)) + \ell(k), \quad (\forall)\ k \in \mathbb{Z}[1,T], \\ \\ x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T), \end{cases} \tag{8}$$

where $\ell : \mathbb{Z}[1,T] \to \mathbb{R}$ is with $\sum_{k=1}^{T} \ell(k) = 0$. More exactly, in Theorem 2.2.15 we prove that problem (8) has at least one solution if there exists a constant $M \geq 0$ such that

$$F(k,t) \leq M, \quad (\forall)\ t \in \mathbb{R} \text{ and } k \in \mathbb{Z}[1,T]$$

and

$$\sum_{k=1}^{T} \limsup_{|t|\to\infty} F(k,t) < \sum_{k=1}^{T} F(k,\tilde{x}_0(k)),$$

where $\tilde{x}_0$ is the unique solution in $X_{P,0} := \left\{\tilde{x} \in X_P \;\middle|\; \sum_{k=1}^{T} \tilde{x}(k) = 0\right\}$ of the simple periodic problem

$$\begin{cases} -\Delta_{p(k-1)}x(k-1) = \ell(k), \quad (\forall)\ k \in \mathbb{Z}[1,T], \\ \\ x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T). \end{cases}$$

A similar result is obtained in the Neumann case (see Theorem 2.2.20). Two concrete examples are provided in order to illustrate the above general existence theorems (see Examples 2.2.11 and 2.2.17).

Next, existence results for problems (4) and (5) with f having concave potential are obtained in Theorems 2.2.21 – 2.2.24. Finally, the solvability of problems (4) and (5) is established in the presence of well ordered lower and upper solutions in Theorem 2.2.28 and Theorem 2.2.33.

In Section 2.3, setting

$$\mathcal{A}_k(x) := -\Delta_{p(k-1)}x(k-1) + r(k)h_{p(k)}(x(k)), \quad (k \in \mathbb{Z}[1,T]),$$

we first deal with multiplicity of solutions for the periodic problem

$$\begin{cases} \mathcal{A}_k(x) = \lambda f(k,x(k)), \quad (\forall)\ k \in \mathbb{Z}[1,T], \\ \\ x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T) \end{cases} \tag{9}$$

and for the Neumann problem

$$\begin{cases} \mathcal{A}_k(x) = \lambda f(k, x(k)), \quad (\forall)\ k \in \mathbb{Z}[1,T], \\ \\ \Delta x(0) = 0 = \Delta x(T), \end{cases} \tag{10}$$

where $r : \mathbb{Z}[1,T] \to (0,\infty)$ is a given function, λ is a positive parameter and f is continuous. Precisely, we provide sufficient conditions ensuring the existence of at least three solutions for suitable values of the parameter. So, in Theorem 2.3.3 and Theorem 2.3.6, under an asymptotic condition on the potential F, we obtain the existence of an open interval Λ_h, such that for any $\lambda \in \Lambda_h$, the problems (9) and (10) admit at least three solutions whose norms are bounded uniformly with respect to $\lambda \in \Lambda_h$. On the other hand, if f is positive on $\mathbb{Z}[1,T] \times [0,\infty)$ then, without asymptotic assumptions on F, we derive the existence of at least three positive solutions for each λ in a well-defined open interval – see Theorem 2.3.8 and Theorem 2.3.11. The main tools in obtaining such results will be two abstract three critical points theorems due to G. Bonanno and P. Candito [28], [31].

Next, if there exists some $\xi > 0$ such that $f(k,\cdot) > 0$ on $(0,\xi)$, $f(k,\xi) = 0$ for all $k \in \mathbb{Z}[1,T]$, and

$$\lim_{t \to 0} \frac{f(k,t)}{|t|^{p(k)-1}} = 0, \quad (\forall)\ k \in \mathbb{Z}[1,T],$$

then we show by a mountain pass type argument that problems (9) and (10) have at least two positive solutions, for sufficiently large values of the parameter λ (see Theorem 2.3.12 and Theorem 2.3.16).

Finally, we study the existence of nontrivial solutions for the one - parameter periodic [Neumann] problem

$$\begin{cases} \mathcal{A}_k(x) = f(k, x(k)) + \lambda b(k) h_{q(k)}(x(k)), \quad (\forall)\ k \in \mathbb{Z}[1,T], \\ x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T)\ [\Delta x(0) = 0 = \Delta x(T)], \end{cases} \tag{11}$$

where $q : \mathbb{Z}[1,T] \to (1,\infty)$, $b : \mathbb{Z}[1,T] \to (0,\infty)$ are given functions and λ is a positive parameter. By adapting some ideas from C. Bereanu *et al.* [13], if there are constants $\theta > \max_{k \in \mathbb{Z}[0,T]} p(k)$ and $\rho > 0$ such that the Ambrosetti-Rabinowitz type condition

$$0 < \theta F(k,t) \le t f(k,t), \quad (\forall)\ k \in \mathbb{Z}[1,T],\ (\forall)\ t \in \mathbb{R} \text{ with } |t| > \rho$$

holds true and assuming a sign asymptotic behavior of the primitive $F(k,\cdot)$ near 0, we prove that problem (11) has at least two nontrivial solutions, for small enough values of the parameter λ; see Theorem 2.3.20 and Theorem

2.3.21. To illustrate the general results in this section, various concrete examples are provided (see Examples 2.3.5, 2.3.10, 2.3.15 and 2.3.23).

It is worth to point out that the variable exponent p must be T-periodic, whenever we refer to the periodic problems in Sections 2.2 and 2.3. Also, we note that we treat in detail the periodic case and we restrict ourselves to only point out the corresponding adaptations for the treatment of the Neumann problems, since the arguments used in the proofs are quite similar. The results in Section 2.2 are proved in [18] and the ones from Section 2.3 in [34] and [123]. We emphasize that, in the periodic case, Theorems 2.2.7, 2.2.9, 2.2.15, 2.2.28, 2.3.3 and 2.3.12 recover and generalize the corresponding ones obtained in [27] and [84] for $p =$ constant.

Chapter 3 focuses on the study of the existence and multiplicity of radial solutions for some Dirichlet and Neumann problems involving the operator $v \mapsto \operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right)$. The first two sections of the chapter are essentially motivated by the existence of classical radial solutions for the nonlinear Dirichlet problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + g(|x|, v) = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R), \quad (12)$$

where $R > 0$, $\mathcal{B}(R) = \{x \in \mathbb{R}^N : |x| < R\}$ $(N \geq 1)$, $|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^N$ and $g : [0, R] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

When dealing with the radial solutions for (12), we are led to study (setting $r = |x|$ and $v(x) = u(r)$) the mixed boundary value problem

$$\left(r^{N-1}\phi(u')\right)' + r^{N-1}g(r, u) = 0, \qquad u'(0) = 0 = u(R), \quad (13)$$

where

$$\phi(y) = \frac{y}{\sqrt{1-y^2}} \quad (y \in (-1, 1)). \quad (14)$$

Actually, instead of the particular ϕ given in (14), we shall consider problem (13) with a general increasing homeomorphism $\phi : (-\eta, \eta) \to \mathbb{R}$ with $\phi(0) = 0$. Following [22], this type of ϕ is called *singular*. In the last decade an increasing number of papers were devoted to boundary value problems involving the singular ϕ-Laplacian operator (we refer the reader to [8]-[15], [19], [20], [22], [23], [36], [49], [50], [99], [102], [103]). These concern with qualitative aspects, as existence or multiplicity of solutions, but effective techniques to compute the solutions seems lagging behind.

In the first section of the chapter we prove the existence of extremal (minimal and maximal) solutions for problem (13) and develop a numerical algorithm for theirs approximation. More exactly, first we apply the monotone iterative technique coupled with the method of lower and upper solutions in order to obtain two monotone sequences that uniformly converge to the extremal solutions of problem (13) (Theorem 3.1.5). Next, if ϕ is an increasing diffeomorphism with $\phi(0) = 0$ and $\phi'(y) \geq d > 0$ for all $y \in (-\eta, \eta)$, we find the approximate extremal solutions of problem (13) (see Algorithm B). With this aim, adopting the same strategy relaying on a shooting technique combined with the classical Euler's method as in P. Jebelean and C. Popa [82], we design a numerical convergent algorithm (Algorithm A) for a problem of type

$$-(r^{N-1}\phi(u'))' = r^{N-1}\ell(r), \qquad u'(0) = 0 = u(R),$$

with $\ell : [0, R] \to \mathbb{R}$ continuous. Also, we provide numerical experiments confirming the theoretical aspects of the section.

In Section 3.2 we deal with the mixed boundary value problem

$$[r^{N-1}\phi(u')]' = r^{N-1}[\mu(r)u^{q-1} - \lambda b(r, u)] \text{ in } [0, R], \quad u'(0) = 0 = u(R), \tag{15}$$

where $\lambda > 0$ is a real parameter and $\phi := \Phi' : (-\eta, \eta) \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$; the continuous function $\Phi : [-\eta, \eta] \to \mathbb{R}$ is of class C^1 on $(-\eta, \eta)$ and $\Phi(0) = 0$. The number $q > 1$ is fixed, $\mu : [0, R] \to \mathbb{R}$ is continuous, positive on $(0, R)$ and the function $b : [0, R] \times [0, A] \to \mathbb{R}$ is continuous, positive on $(0, R) \times (0, A)$ and satisfies $b(r, 0) = 0 = b(r, A)$ for all $r \in [0, R]$. Note that (13) is nothing else but a problem of type (15) (one takes $\Phi(s) = 1 - \sqrt{1 - s^2}$), with the λ-parameterized nonlinearity $g(r, u) = \lambda b(r, u) - \mu(r)u^{q-1}$. Using Szulkin's critical point theory, we provide sufficient conditions ensuring the existence of at least one or at least two nontrivial solutions, for large enough values of the parameter.

As a first step, with $f : [0, R] \times \mathbb{R} \to [0, \infty)$ defined by

$$f(r, s) = \begin{cases} 0, & \text{if } s < 0 \text{ or } s > A, \\ b(r, s), & \text{if } s \in [0, A], \end{cases}$$

we show in Proposition 3.2.1 that if u is a solution of problem

$$[r^{N-1}\phi(u')]' = r^{N-1}[\mu(r)|u|^{q-2}u - \lambda f(r, u)] \text{ in } [0, R], \quad u'(0) = 0 = u(R), \tag{16}$$

then $0 \leq u \leq A$ and hence, u solves problem (15). Next, a variational approach is introduced for problem (16). Thus, we consider the set

$$K_0 := \{u \in W^{1,\infty}(0, R) : \|u'\|_\infty \leq \eta, \ u(R) = 0\},$$

where $\| \cdot \|_\infty$ denotes the usual supremum norm on $C[0, R]$. The energy functional $I_\lambda : C[0, R] \to (-\infty, +\infty]$ associated to problem (16) will be

$$I_\lambda(u) = \begin{cases} \displaystyle\int_0^R r^{N-1}\Phi(u')\,dr + \int_0^R r^{N-1}\left[\frac{\mu(r)}{q}|u|^q - \lambda F(r,u)\right] dr, \text{if } u \in K_0, \\ \\ +\infty, \quad \text{otherwise,} \end{cases}$$

where $F(r, s) = \int_0^s f(r, \xi)d\xi$ for all $r \in [0, R]$ and $s \in \mathbb{R}$. We prove in Lemma 3.2.3 that each critical point u of I_λ (in the sense of Szulkin) is a solution of (16), hence u solves problem (15).

The main result in this section is Theorem 3.2.4 – showing that under the above assumptions on the data, problem (15) has at least one nontrivial solution, for sufficiently large values of the parameter λ. If in addition $\min_{[0,R]} \mu > 0$ and b satisfies

$$\lim_{s\to 0+} \frac{b(r, s)}{s^{q-1}} = 0 \qquad \text{uniformly with } r \in [0, R],$$

then (15) has at least two nontrivial solutions for all λ large enough.

We also note that the results in Theorem 3.2.4 are of the same type with those obtained in Theorem 2.3.12 and Theorem 2.3.16 for the periodic and Neumann problems involving the discrete $p(\cdot)$-Laplacian. Some examples of applications conclude the section.

The results in Section 3.1 are proved in [83] and the ones from Section 3.2 are obtained in [17].

In the last section of the chapter, we study the existence of positive solutions for Neumann problems with singular attractive restoring force:

$$(r^{N-1}\phi(u'))' + r^{N-1}f(u) = r^{N-1}h(r) \text{ in } [R_1, R_2], \quad u'(R_1) = 0 = u'(R_2) \tag{17}$$

and with singular repulsive restoring force:

$$(\phi(u'))' - f(u) = h(r) \text{ in } [R_1, R_2], \quad u'(R_1) = 0 = u'(R_2), \tag{18}$$

where $N \geq 1$ is an integer, $0 \leq R_1 < R_2$, functions $h : [R_1, R_2] \to \mathbb{R}$ and $f : (0, +\infty) \to \mathbb{R}$ are continuous and ϕ is an increasing homeomorphism with $\phi(0) = 0$. With ϕ given in (14), solutions of (17) correspond to the classical radial ones of the Neumann problem

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}}\right) + f(v) = h(|x|) \text{ in } \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\mathcal{A},$$

where $\mathcal{A}$ is the annular domain $\{x \in \mathbb{R}^N : R_1 \leq |x| \leq R_2\}$ and $v(x) = u(r)$ with $r = |x|$. As usual, $\frac{\partial v}{\partial \nu}$ denotes the outward normal derivative of v.

If there exists a constant $\alpha > 0$ such that $f(\alpha) \geq \|h\|_\infty$, $R_1 > 0$ and

$$\limsup_{u \to +\infty} f(u) < \overline{h},$$

then using the method of lower and upper solutions, we obtain in Theorem 3.3.3 that problem (17) has at least one positive solution $u \geq \alpha$. Here, we have denoted

$$\overline{h} = \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} r^{N-1} h(r) dr.$$

In Theorem 3.3.10, using the Leray-Schauder degree, we prove that problem (18) is solvable if f satisfies the following conditions:

$$\liminf_{u \to 0+} \left(f(u) + \min\{0, \overline{h}\} \right) > 0, \quad \limsup_{u \to +\infty} f(u) < -\overline{h},$$

$$\liminf_{u \to +\infty} f(u) > -\infty \quad \text{and} \quad \int_0^1 f(u) du = +\infty.$$

The proofs of the above two existence theorems rely on some ideas originated in P. Jebelean and J. Mawhin [78]. Also, we note that for $N \geq 2$, in the repulsive case, the Neumann problem

$$(r^{N-1}\phi(u'))' - r^{N-1} f(u) = r^{N-1} h(r) \text{ in } [R_1, R_2], \quad u'(R_1) = 0 = u'(R_2)$$

still remains an open one.

As above mentioned in the summary of each chapter, most of the results presented in this thesis are part of the following publications:

- C. Bereanu, P. Jebelean and C. Şerban, *Ground state and mountain pass solutions for discrete $p(\cdot)$-Laplacian*, Bound. Value Probl. **104:2012** (2012).
- C. Bereanu, P. Jebelean and C. Şerban, *Nontrivial solutions for a class of one-parameter problems with singular ϕ-Laplacian*, Ann. Univ. Buchar. Math. Ser. **3 (LXI)** (2012), 155-162.
- C. Bereanu, P. Jebelean and C. Şerban, *Periodic and Neumann problems for discrete $p(\cdot)$-Laplacian*, J. Math. Anal. Appl. **399** (2013), 75-87.
- G. Bonanno, P. Jebelean and C. Şerban, *Three solutions for discrete anisotropic periodic and Neumann problems*, Dynam. Systems Appl. **22** (2013), 183-195.
- P. Jebelean, C. Popa and C. Şerban, *Numerical extremal solutions for a mixed problem with singular ϕ-Laplacian*, Nonlinear Differ. Equ. Appl., DOI: 10.1007/s00030-013-0247-9.

• P. Jebelean and C. Şerban, *Ground state periodic solutions for difference equations with discrete p-Laplacian*, Appl. Math. Comput. **217** (2011), 9820-9827.

• C. Şerban, *Existence of solutions for discrete p-Laplacian with potential boundary conditions*, J. Difference Equ. Appl. **19 (3)** (2013), 527-537.

• C. Şerban, *Multiplicity of solutions for periodic and Neumann problems involving the discrete $p(\cdot)$-Laplacian*, Taiwanese J. Math. **17 (4)** (2013), 1425-1439.

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1

Preliminaries

In this chapter we list some preliminary notions and results which are needed in the sequel.

Definitions and basic properties

Let $(Y, \|\cdot\|)$ be a real Banach space. A *minimizing sequence* for a function $\varphi : Y \to (-\infty, +\infty]$ is a sequence $\{u_n\} \subset Y$ such that

$$\lim_{n\to\infty} \varphi(u_n) = \inf_{u\in Y} \varphi(u).$$

It is clear that in a finite dimensional space, existence of a convergent minimizing sequence is equivalent with the existence of a bounded minimizing sequence. A function $\varphi : Y \to (-\infty, +\infty]$ will be called *lower semi-continuous* (l.s.c for short), if

$$\varphi(u) \leq \liminf_{n\to\infty} \varphi(u_n), \tag{1.1}$$

for every sequence $\{u_n\} \subset Y$ converging strongly to $u \in Y$. If the inequality (1.1) holds true for any sequence $\{u_n\} \subset Y$ converging weakly to $u \in Y$, φ is said to be *weakly lower semi-continuous* (w.l.s.c for short). Note that the sum of two l.s.c (resp. w.l.s.c) functions is l.s.c (resp. w.l.s.c).

Theorem 1.1 (Theorem 1.1 in [105])**.** *Let φ be w.l.s.c on a reflexive Banach space Y which has a bounded minimizing sequence. Then, φ has a minimum on Y.*

The existence of a bounded minimizing sequence will be in particular insured when φ is *coercive*, i.e.,

$$\varphi(u) \to +\infty \quad \text{as} \quad \|u\| \to \infty.$$

Therefore, by Theorem 1.1, we have the following well-known result from calculus of variations

Theorem 1.2 (Proposition 1.2, page 135 in [88]). *If φ is w.l.s.c and coercive on a reflexive Banach space Y, then φ is bounded from below and* $\inf \varphi$ *is attained at a point $u \in Y$.*

The function $\varphi : Y \to (-\infty, +\infty]$ is *convex* if

$$\varphi(\lambda u + (1-\lambda)v) \leq \lambda\varphi(u) + (1-\lambda)\varphi(v) \tag{1.2}$$

holds true for every $\lambda \in (0,1)$ and all $u, v \in Y$. The function φ is called *strictly convex* if the inequality in (1.2) is strict whenever $\varphi(u) < +\infty$, $\varphi(v) < +\infty$, $u \neq v$ and $\lambda \in (0,1)$. The function φ is said to be *(strictly) concave* if $-\varphi$ is (strictly) convex. It is known that if φ is l.s.c and convex, then it is w.l.s.c [105, Theorem 1.2].

A function $\varphi : Y \to (-\infty, +\infty]$ is *proper* if

$$D(\varphi) := \{u \in Y : \ \varphi(u) < +\infty\} \neq \emptyset.$$

The set $D(\varphi)$ is called *the effective domain* of φ. Given a subset K of Y, the function I_K, defined by

$$I_K(u) = \begin{cases} 0, & \text{if } u \in K \\ +\infty, & \text{if } u \in Y \setminus K \end{cases}$$

is called *the indicator function* of K. Clearly, if $K \subset Y$ is nonempty, closed and convex, then I_K is proper, convex and l.s.c.

We conclude this paragraph with the following two elementary and intuitive results.

Proposition 1.3 (Proposition 2.20 in [6]). *Any convex, proper and l.s.c function is bounded from below by an affine function.*

Proposition 1.4 (Proposition 1.5 in [105]). *Let $G \in C^1(\mathbb{R}, \mathbb{R})$ be a strictly convex function. The following properties are equivalent:*

(i) there exist $c \in \mathbb{R}$ such that $G'(c) = 0$;

(ii) $G(t) \to +\infty$ when $|t| \to \infty$.

Classical critical point theory

Many equations are of the form $\varphi'(u) = 0$ in an appropriate Banach space Y. The equation $\varphi'(u) = 0$ is said to be *the Euler-Lagrange equation* of the energy functional $\varphi : Y \to \mathbb{R}$. Its solutions are understood in the sense

$$\langle \varphi'(u), v\rangle = 0, \quad (\forall)\ v \in Y$$

and are precisely the *critical points* of φ. If $\varphi : Y \to \mathbb{R}$ is Gâteaux differentiable at a local minimum point $u \in Y$, then u is a critical point of φ. The value $\varphi(u)$ is called *critical value* of φ if u is a critical point.

Over the last decades, critical point theory has been applied to various problems in differential, partial differential and difference equations, mathematical physics and geometry. Historically, modern critical point theory goes back to the works of M. Morse, L. Ljusternik and L. Schnirelmann (1934), where the critical values of a functional $\varphi : Y \to \mathbb{R}$ were characterized as inf max values over a class of sets in the space Y.

Further, in the joint paper [113] (1964) devoted to an extension of Morse theory, R. Palais and S. Smale have introduced a compactness condition, namely condition (C), which is the first form of the so called *Palais-Smale condition* (in short *(PS) condition*). This condition was a base for the modern development of critical point theory being intensively used in many arguments related to the existence of critical points. We recall the (PS) condition as it appears in the book of J.T. Schwartz [120] (1969):

Definition 1.5. A C^1 functional $\varphi : Y \to \mathbb{R}$ satisfies the (PS) condition if any sequence $\{u_n\} \subset Y$ such that $\{\varphi(u_n)\}$ is bounded and

$$\varphi'(u_n) \to 0, \quad \text{as } n \to \infty$$

has a convergent subsequence.

From (PS) condition, it follows that the set of critical points for a bounded functional is compact. Other variants of the (PS) condition were introduced by H. Brézis *et al.* [35], G. Cerami [44] and A. Szulkin [121]. We refer to the recent paper of J. Mawhin and M. Willem [106] for a detailed study on the origin and evolution of the (PS) condition and its variants in critical point theory.

In 1973, in the seminal paper of A. Ambrosetti and P.H. Rabinowitz [5], it was formulated the celebrated *Mountain Pass Theorem* which is one of the most powerful tools in Nonlinear Analysis used to prove the existence of critical points of energy functionals. The concept was further developed by L. Nirenberg, H. Brézis, I. Ekeland, J. Mawhin, M. Willem, A. Szulkin and many other authors in different directions.

Thus, the Mountain Pass Theorem has numerous generalizations and during the years it has been used in the treatment of various classes of boundary value problems. The reader is referred to monograph of Y. Jabri

[75] for a survey on some variants and interesting applications of this abstract result. Also, we note that the original proof of the Mountain Pass Theorem of Ambrosetti and Rabinowitz is based on a *Deformation Lemma* due to D.C. Clark [48]. We recall the theorem as it appears in the work of P.H. Rabinowitz [116].

Theorem 1.6 (Theorem 2.2 in [116])**.** *Let* $(Y, \|\cdot\|)$ *be a real Banach space and* $\varphi \in C^1(Y, \mathbb{R})$ *satisfying the (PS) condition. Suppose* $\varphi(0) = 0$ *and*

(i) there exist constants $\alpha, \rho > 0$ *such that* $\varphi(u) \geq \alpha$ *if* $\|u\| = \rho$*;*

(ii) $\varphi(e) \leq 0$ *for some* $e \in Y$*, with* $\|e\| > \rho$*.*

Then, φ *has a critical value* $c \geq \alpha$ *which can be characterized as*

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1]; Y) \ : \ \gamma(0) = 0,\ \gamma(1) = e\}$*.*

Geometrically, when $Y = \mathbb{R}^2$ the above assumptions (i) and (ii) mean that the origin lies in a 'valley' surrounded by a 'mountain'

$$\Gamma_\varphi = \{(u, \varphi(u)) \in \mathbb{R}^3 \ : \ u \in \mathbb{R}^2\}$$

and the conclusion of the theorem tells us that there exists a 'mountain pass' joining $(0,0)$ and $(e, \varphi(e))$ that contains a critical value. More intuitive, let 0 and e be two cities separated by a mountains chain. If $\varphi(u)$ represents the altitude at a point u, then there exists a road between the two cities with a minimal altitude, i.e., in other words, a 'mountain pass'.

Theorem 1.7 (Theorem 2.7 in [116])**.** *Let* Y *be a real Banach space and* $\varphi \in C^1(Y, \mathbb{R})$ *satisfying the (PS) condition. If* φ *is bounded from below, then*

$$c \equiv \inf_Y \varphi$$

is a critical value of φ*.*

Next, we recall the *Saddle Point Theorem* (see e.g., [116, Theorem 4.6], [105, Theorem 4.7]), which is known as another important tool in critical point theory.

Theorem 1.8. *Let* Y *be a real Banach space and* $\varphi \in C^1(Y, \mathbb{R})$*. Assume that* Y *splits into a direct sum of two subspaces* $Y = Y^- \oplus Y^+$ *with*

$$dim\ Y^- < \infty$$

and there exists $R > 0$ such that

$$\sup_{S_R} \varphi < \inf_{Y^+} \varphi,$$

where $S_R = \{u \in Y^- : |u| = R\}$ ($|\cdot|$ is the norm in Y^-). Let

$$B_R^- = \{u \in Y^- : |u| \leq R\},$$

$$G = \{g \in C(B_R^-, Y) : g(s) = s \text{ if } s \in S_R\}$$

and

$$c = \inf_{g \in G} \max_{s \in B_R^-} \varphi(g(s)).$$

Then, if φ satisfies the (PS) condition, c is a critical value of φ.

Also, we shall need the following result.

Proposition 1.9. *Let $(Y, \|\cdot\|)$ be a real Banach space and $\varphi \in C^1(Y, \mathbb{R})$ satisfying the (PS) condition. If there exists an open set U such that*

$$-\infty < \inf_{\overline{U}} \varphi < \inf_{\partial U} \varphi, \tag{1.3}$$

then φ has at least one critical point $u \in U$ such that $\varphi(u) = \inf_{\overline{U}} \varphi$.

Proof. Let $c_0 = \inf_{\overline{U}} \varphi$ and $\{\varepsilon_n\}$ be a sequence with $\varepsilon_n \to 0$ and

$$0 < \varepsilon_n < \inf_{\partial U} \varphi - c_0, \quad (\forall)\ n \in \mathbb{N}. \tag{1.4}$$

Using *Ekeland's variational principle* [55], applied to $\varphi|_{\overline{U}}$, for each $n \in \mathbb{N}$, there exists $v_n \in \overline{U}$ such that

$$\varphi(v_n) \leq c_0 + \varepsilon_n \tag{1.5}$$

and

$$\varphi(v) \geq \varphi(v_n) - \varepsilon_n \|v - v_n\|, \quad (\forall)\ v \in \overline{U}. \tag{1.6}$$

From (1.4) and (1.5) it follows $\varphi(v_n) < \inf_{\partial U} \varphi$, which ensures that $v_n \in U$, for all $n \in \mathbb{N}$. Let $h \in Y$ with $\|h\| = 1$, $n \in \mathbb{N}$ be arbitrarily chosen and $t_0 := t_0(h, n) \in (0, 1)$ be so that $v_n + th \in U$, for all $t \in (0, t_0)$. Using (1.6), we get

$$\varphi(v_n + th) - \varphi(v_n) \geq -\varepsilon_n t \|h\| = -\varepsilon_n t.$$

Dividing by t and letting $t \to 0^+$, one obtains

$$\langle \varphi'(v_n), h \rangle \geq -\varepsilon_n, \quad (\forall)\ h \in Y \text{ with } \|h\| = 1. \tag{1.7}$$

If we change h with $-h$ in (1.7), one has

$$\langle \varphi'(v_n), h\rangle \leq \varepsilon_n$$

and hence

$$\|\varphi'(v_n)\| \leq \varepsilon_n. \tag{1.8}$$

On the other hand, from (1.5) it is clear that

$$\varphi(v_n) \to c_0. \tag{1.9}$$

Since φ satisfies the (PS) condition, (1.8) and (1.9) ensure that $\{v_n\}$ contains a subsequence, still denoted by $\{v_n\}$, convergent to some $u \in \overline{U}$. Also, from (1.8) and (1.9), we infer that $\varphi(u) = c_0$ and $\varphi'(u) = 0$. Then, by virtue of (1.3), $u \in U$ and the proof is complete. ■

For $\sigma > 0$, we shall denote $B_\sigma = \{v \in Y : \|v\| < \sigma\}$ and by $\overline{B}_\sigma$ its closure.

Proposition 1.10. *Let $(Y, \|\cdot\|)$ be a real Banach space and $\varphi \in C^1(Y, \mathbb{R})$. Suppose that φ satisfies (PS) condition together with*

(i) $\varphi(0) = 0$ and there exists $\rho > 0$ such that

$$-\infty < \inf_{\overline{B}_\rho} < 0 < \inf_{\partial B_\rho} \varphi; \tag{1.10}$$

(ii) $\varphi(e) \leq 0$ for some $e \in Y \setminus \overline{B}_\rho$.

Then, φ has at least two nontrivial critical points.

Proof. From Theorem 1.6 there exists a first nontrivial critical point $u_1 \in Y$ with $\varphi(u_1) > 0$. On the other hand, using Proposition 1.9 with $U = B_\rho$ and (1.10), it follows that $\inf_{\overline{B}_\rho} \varphi$ is a critical value of φ. This implies the existence of a second critical point u_2 with $\varphi(u_2) < 0$. Note that u_2 is nontrivial and different from u_1 because $\varphi(0) = 0$ and $\varphi(u_1) > 0$. ■

Remark 1.11. (i) It is a simple matter to check that if in Proposition 1.10 we assume, in addition, that φ is even, then it has at least four nontrivial critical points.

(ii) We note that Propositions 1.9 and 1.10 are consequences of Proposition 1, respectively Proposition 2 proved in [13] for Szulkin's type functionals (see Remark 1.16 below). However, we have chose to give direct proofs in the frame of the classical theory.

Szulkin's critical point theory

Let $(Y, \|\cdot\|)$ be a real Banach space and $I : Y \to (-\infty, +\infty]$ be a functional of the type

$$I = \Phi + \psi, \tag{1.11}$$

where $\Phi \in C^1(Y; \mathbb{R})$ and ψ is proper, convex and l.s.c. In 1986, A. Szulkin [121] generalize for functionals of type (1.11) results from the classical critical point theory. Accordingly, a point $u \in Y$ is said to be *a critical point* of I if it satisfies the variational inequality:

$$\langle \Phi'(u), v - u \rangle + \psi(v) - \psi(u) \geq 0, \quad (\forall)\ v \in Y. \tag{1.12}$$

A number $c \in \mathbb{R}$ such that $I^{-1}(c)$ contains a critical point is called *a critical value* of the functional I.

Proposition 1.12 (Proposition 1.1 in [121])**.** *If I satisfies* (1.11)*, each local minimum point of I is a critical point of I.*

Definition 1.13. The functional I is said to satisfy the (PS) condition if every sequence $\{u_n\} \subset Y$ for which $I(u_n) \to c \in \mathbb{R}$ and

$$\langle \Phi'(u_n), v - u_n \rangle + \psi(v) - \psi(u_n) \geq -\varepsilon_n \|v - u_n\|, \quad (\forall)\ v \in Y,$$

where $\varepsilon_n \to 0$, possesses a convergent subsequence.

The following theorem extends the classical Mountain Pass Theorem from the previous paragraph.

Theorem 1.14 (Theorem 3.2 in [121])**.** *Suppose that I satisfies* (1.11)*, the (PS) condition and*

(i) $I(0) = 0$ and there exist $\alpha, \rho > 0$ such that $I(u) \geq \alpha$ if $\|u\| = \rho$;

(ii) $I(e) \leq 0$ for some $e \in Y$, with $\|e\| > \rho$.

Then, I has a critical value $c \geq \alpha$ which can be characterized by

$$c = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I(\gamma(t)),$$

where $\Gamma = \{\gamma \in C([0,1]; Y)\ :\ \gamma(0) = 0,\ \gamma(1) = e\}$.

The next result generalize Theorem 1.7 for Szulkin's type functionals.

Theorem 1.15 (Theorem 1.7 in [121])**.** *If I satisfies* (1.11)*, the (PS) condition and is bounded from below, then*

$$c \equiv \inf_Y I$$

is a critical value of I.

Remark 1.16. Observe that if $\psi \equiv 0$, then $I \equiv \Phi \in C^1(Y, \mathbb{R})$ and hence, the above definitions and results yield the corresponding ones from the classical case.

Three critical points theorems

In this paragraph we recall two abstract theorems which will be needed in the proofs of the results in Section 2.3. The first was obtained by G. Bonanno [28] as a consequence of a three critical points theorem of B. Ricceri [118], by using some results on a suitable minimax inequality (also see [29]). The second one was established by G. Bonanno and P. Candito [31] (also see [33]) and it is a finite dimensional variant of Theorem 3.3 in [30].

Theorem 1.17 (Theorem 2.1 in [28])**.** *Let $(Y, \|\cdot\|)$ be a separable and reflexive real Banach space, and let $\psi, J : Y \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals. Assume that there exists $y_0 \in Y$ such that $\psi(y_0) = J(y_0) = 0$ and $\psi(y) \geq 0$ for every $y \in Y$ and that there exist $y_1 \in Y$, $\omega > 0$ such that*

(i_1) $\omega < \psi(y_1)$;

(i_2) $\sup_{\psi(y)<\omega} J(y) < \omega \dfrac{J(y_1)}{\psi(y_1)}$.

Further, put

$$\overline{a} = \frac{h\omega}{\omega\frac{J(y_1)}{\psi(y_1)} - \sup_{\psi(y)<\omega} J(y)},$$

with $h > 1$, assume that the functional $\psi - \lambda J$ is w.l.s.c, satisfies the (PS) condition and

(i_3) $\lim_{\|y\|\to+\infty} (\psi(y) - \lambda J(y)) = +\infty$, *for every* $\lambda \in [0, \overline{a}]$.

Then, there exists an open interval $\Lambda \subseteq [0, \overline{a}]$ and a positive real number μ such that, for each $\lambda \in \Lambda$, the equation $\psi'(y) - \lambda J'(y) = 0$ admits at least three solutions in Y, whose norms are less than μ.

Theorem 1.18 (Theorem 2.1 in [31], Theorem 31 in [33])**.** *Let Y be a finite dimensional real Banach space and $\psi, J : Y \to \mathbb{R}$ be two functionals of class C^1 on Y, with ψ coercive. Moreover, assume that*

(i_4) ψ *is convex and* $\inf_Y \psi = \psi(0) = J(0) = 0$;

(i_5) *for each* $\lambda > 0$ *and every* $u_1,\ u_2$ *which are local minima for the functional* $\psi - \lambda J$ *such that* $J(u_1) \geq 0$ *and* $J(u_2) \geq 0$, *one has*

$$\inf_{\xi \in [0,1]} J(\xi u_1 + (1-\xi) u_2) \geq 0.$$

Further, assume that there are two positive constants $\omega_1,\ \omega_2$ *and* $v \in Y$, *with* $\omega_1 < \psi(v) < \omega_2/2$ *such that*

(i_6) $$\frac{\sup_{y \in \psi^{-1}(-\infty,\omega_1)} J(y)}{\omega_1} < \frac{J(v)}{2\psi(v)} \quad \text{and} \quad \frac{\sup_{y \in \psi^{-1}(-\infty,\omega_2)} J(y)}{\omega_2} < \frac{J(v)}{4\psi(v)}.$$

Then, for each

$$\lambda \in \left(\frac{2\psi(v)}{J(v)}, \min\left\{ \frac{\omega_1}{\sup_{y \in \psi^{-1}(-\infty,\omega_1)} J(y)}, \frac{\omega_2/2}{\sup_{y \in \psi^{-1}(-\infty,\omega_2)} J(y)} \right\} \right),$$

the functional $\psi - \lambda J$ *admits at least three distinct critical points* $y_1,\ y_2,\ y_3$ *such that* $y_1 \in \psi^{-1}(-\infty, \omega_1)$, $y_2 \in \psi^{-1}(\omega_1, \omega_2/2)$ *and* $y_3 \in \psi^{-1}(-\infty, \omega_2)$.

2

Existence and multiplicity of solutions for discrete $p(\cdot)$-Laplacian

Contents

Throughout this chapter, T will be a positive integer, $p : \mathbb{Z}[0,T] \to (1,\infty)$ and for given $s \in (1,\infty)$, h_s will stand for the homeomorphism defined by $h_s(x) = |x|^{s-2}x$, for all $x \in \mathbb{R}$. Here and below, for $a, b \in \mathbb{N}$ with $a < b$, we use the notation $\mathbb{Z}[a,b] := \{a, a+1, \dots, b\}$. Also, we set

$$\mathbb{R}^{\mathbb{Z}[1,T]} := \{\ell : \mathbb{Z}[1,T] \to \mathbb{R}\}.$$

From now on, $\Delta_{p(\cdot)}$ stands for the discrete $p(\cdot)$-Laplacian operator, that is,

$$\begin{aligned}\Delta_{p(k-1)}x(k-1) :&= \Delta(h_{p(k-1)}(\Delta x(k-1))) \\ &= h_{p(k)}(\Delta x(k)) - h_{p(k-1)}(\Delta x(k-1)),\end{aligned} \tag{2.1}$$

where $\Delta x(k) = x(k+1) - x(k)$ is the forward difference operator. Hereafter, we also employ the notations:

$$p^- = \min_{k\in\mathbb{Z}[0,T]} p(k),\ \ p^+ = \max_{k\in\mathbb{Z}[0,T]} p(k) \ \text{ and } \ \underline{p} = \min_{k\in\mathbb{Z}[1,T]} p(k),\ \ \overline{p} = \max_{k\in\mathbb{Z}[1,T]} p(k).$$

We are concerned in this chapter with existence and multiplicity of solutions for discrete $p(\cdot)$-Laplacian equations subjected to periodic, Neumann and general potential type boundary conditions.

Questions from various fields of research, such as computer science, mathematical physics, neural and electrical networks, number theory and statistics, genetics, economics and many others, leads to consideration of nonlinear difference equations. For this reason, the studies regarding discrete boundary value problems has captured a special attention in recent years and so, many authors have approached such type of problems using various techniques. The variational method already known for continuous problems appears as being a very fruitful one. In this direction we mention the papers [2], [3], [27], [31], [32], [40] - [43], [84], [87], [122], [124] - [126] where equations involving the discrete p-Laplacian operator, subjected to classical or less classical boundary conditions, have been considered. Also, we note the recent paper of J. Mawhin [104] where variational techniques are employed to obtain the existence of periodic solutions for systems involving a general discrete ϕ-Laplacian operator.

The research concerning boundary value problems with the discrete $p(\cdot)$-Laplacian operator was initiated by M. Mihăilescu *et al.* in [110], where some eigenvalue problems were investigated. Also, the same authors studied in [111] the existence of homoclinic solutions for a class of non-homogeneous discrete anisotropic equation with periodic coefficients.

Existence and multiplicity results for discrete $p(\cdot)$-Laplacian equations subjected to homogeneous Dirichlet boundary conditions were obtained in the last years by M. Galewski [66], M. Galewski and R. Wieteska [69], B. Koné and S. Ouaro [90], R. Mashiyev *et al.* [100].

Also, the existence of nontrivial solutions for some discrete anisotropic problems was investigated in M. Galewski and S. Głąb [67]. Using mountain pass type arguments and the Karush-Kuhn-Tucker theorem, M. Galewski *et al.* establish in [68] the existence of at least two positive solutions for discrete

$p(\cdot)$-Laplacian operator, in the case of Dirichlet boundary conditions. In G. Molica Bisci and D. Repovš [112], the existence of infinitely many solutions for discrete anisotropic equations are obtained by means of some related variational arguments.

Existence and multiplicity of solutions for discrete $p(\cdot)$-Laplacian problems with other boundary conditions were investigated by very few authors. Actually, to the best of our knowledge, only A. Guiro *et al.* [72] and B. Koné and S. Ouaro [91] proved existence and uniqueness of solutions for a family of discrete anisotropic Neumann and mixed boundary value problems, respectively. We point out that, in the Neumann case, the results and the setting in [72] are of different type from those considered in Sections 2.2 and 2.3. Hence, an objective of the present chapter is to partly fill this gap.

The rest of the chapter is organized as follows. The existence of ground state and mountain pass type solutions for discrete $p(\cdot)$-Laplacian equations subjected to a general potential type boundary condition is established in Section 2.1. In Section 2.2, we present existence results for some periodic and Neumann problems, while Section 2.3 is devoted to the multiplicity of solutions for periodic and Neumann problems.

2.1 Existence results for problems with a general potential boundary condition

In this section we deal with equations of type

$$-\Delta_{p(k-1)}x(k-1) = f(k, x(k)), \quad (\forall)\ k \in \mathbb{Z}[1, T], \tag{2.1.1}$$

subjected to the potential boundary condition

$$(h_{p(0)}(\Delta x(0)), -h_{p(T)}(\Delta x(T))) \in \partial j(x(0), x(T+1)), \tag{2.1.2}$$

where $f : \mathbb{Z}[1, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $j : \mathbb{R} \times \mathbb{R} \to (-\infty, +\infty]$ is convex, proper, l.s.c. and ∂j denotes the subdifferential of j. Recall, for $z \in \mathbb{R} \times \mathbb{R}$ the set $\partial j(z)$ is defined by

$$\partial j(z) = \{\zeta \in \mathbb{R} \times \mathbb{R} :\ j(\xi) - j(z) \geq (\zeta|\xi - z),\ (\forall)\ \xi \in \mathbb{R} \times \mathbb{R}\},$$

where $(\cdot|\cdot)$ stands for the usual inner product in $\mathbb{R} \times \mathbb{R}$.

We point out that the potential boundary condition (2.1.2) recovers the classical ones. For instance, denoting by I_K the indicator function of a closed, nonempty and convex set $K \subset \mathbb{R} \times \mathbb{R}$, the Dirichlet and Neumann boundary conditions are obtained by choosing $j = I_K$ with $K = \{(0, 0)\}$

in the Dirichlet case, respectively $K = \mathbb{R} \times \mathbb{R}$ for the Neumann case. Also, if p is T-periodic, taking $K = \{(x, x),\ x \in \mathbb{R}\}$ ($K = \{(x, -x),\ x \in \mathbb{R}\}$) and $j = I_K$ we get the periodic (antiperiodic) conditions; see Remark 2.1.15 below for more details and other choices of j (also, see L. Gasinski and N.S. Papageorgiou [70] and P. Jebelean [76]). So, among others, an unified approach of all these classical boundary conditions is provided.

The main ingredient here will be Szulkin's critical point theory [121]. We obtain the existence of solutions in a coercive case as well as the existence of nontrivial solutions when the corresponding energy functional has a 'mountain pass' geometry. In order to prove these existence results, we shall employ some ideas originated in P. Jebelean and G. Moroşanu [80] (also, see P. Jebelean [76]), combined with specific technicalities due to the discrete and anisotropic character of the problem.

The functional framework and the variational approach for problem (2.1.1), (2.1.2) are presented in the first paragraph. In the second one, we obtain the existence of ground state type solutions, while Paragraph 2.1.3 provides existence results for mountain pass type solutions.

The main results in this section are proved in [16] and recover and generalize the similar ones for $p =$ constant obtained in [122].

2.1.1 The functional framework

We consider the finite dimensional space

$$X_{PO} := \{x : \mathbb{Z}[0, T+1] \to \mathbb{R}\}$$

and for $x \in X_{PO}$ and $\eta > 0$, we introduce

$$\mathcal{N}_{x,\eta}(\nu) := \sum_{k=1}^{T+1} \frac{1}{p(k-1)} \left| \frac{\Delta x(k-1)}{\nu} \right|^{p(k-1)} + \eta \sum_{k=1}^{T} \frac{1}{p(k)} \left| \frac{x(k)}{\nu} \right|^{p(k)}, \quad (\nu > 0).$$

Proposition 2.1.1. *For any $\eta > 0$, the functional $\|\cdot\|_{\eta,p(\cdot)} : X_{PO} \to \mathbb{R}$, defined by*

$$\|x\|_{\eta,p(\cdot)} = \inf\{\nu > 0 \ : \ \mathcal{N}_{x,\eta}(\nu) \le 1\}$$

is a norm (of Luxemburg type) on X_{PO}.

Proof. If $\|x\|_{\eta,p(\cdot)} = 0$, there exists a sequence $\{\nu_n\} \subset (0, 1)$ such that $\nu_n \to 0^+$ and $\mathcal{N}_{x,\eta}(\nu_n) \le 1$. Hence, one has

$$\frac{1}{\nu_n^{p^-}} \mathcal{N}_{x,\eta}(1) \le \mathcal{N}_{x,\eta}(\nu_n) \le 1, \quad (\forall)\ n \in \mathbb{N}.$$

Then, letting $n \to \infty$ in $\mathcal{N}_{x,\eta}(1) \le \nu_n^{p^-}$, we get $\mathcal{N}_{x,\eta}(1) = 0$, yielding $x = 0$.

Further, for $x \in X_{PO}$ and $\alpha \in \mathbb{R} \setminus \{0\}$, we have

$$\|\alpha x\|_{\eta,p(\cdot)} = \inf\{\nu > 0 \;:\; \mathcal{N}_{\alpha x,\eta}(\nu) \le 1\} = \inf\{\nu > 0 \;:\; \mathcal{N}_{x,\eta}(\nu/|\alpha|) \le 1\}$$
$$\underset{\nu^\star := \nu/|\alpha|}{=} \inf\{|\alpha|\nu^\star > 0 \;:\; \mathcal{N}_{x,\eta}(\nu^\star) \le 1\} = |\alpha|\|x\|_{\eta,p(\cdot)}.$$

To show that

$$\|x+y\|_{\eta,p(\cdot)} \le \|x\|_{\eta,p(\cdot)} + \|y\|_{\eta,p(\cdot)}, \tag{2.1.3}$$

for $x, y \in X_{PO}$, let $\varepsilon > 0$. From the definition of $\|\cdot\|_{\eta,p(\cdot)}$, there exist $\nu_1,\ \nu_2 > 0$ with $\mathcal{N}_{x,\eta}(\nu_{1,2}) \le 1$ and $\nu_1 < \|x\|_{\eta,p(\cdot)} + \varepsilon$, $\nu_2 < \|y\|_{\eta,p(\cdot)} + \varepsilon$. Using the convexity of the function $t \mapsto |t|^\alpha$ ($\alpha > 1$), we obtain

$$\begin{aligned}
\mathcal{N}_{x+y,\eta}(\nu_1+\nu_2) &= \sum_{k=1}^{T+1} \frac{1}{p(k-1)} \left| \frac{\Delta x(k-1) + \Delta y(k-1)}{\nu_1+\nu_2} \right|^{p(k-1)} \\
&\quad + \eta \sum_{k=1}^{T} \frac{1}{p(k)} \left| \frac{x(k)+y(k)}{\nu_1+\nu_2} \right|^{p(k)} \\
&= \sum_{k=1}^{T+1} \frac{1}{p(k-1)} \left| \frac{\nu_1}{\nu_1+\nu_2} \frac{\Delta x(k-1)}{\nu_1} + \left(1 - \frac{\nu_1}{\nu_1+\nu_2}\right) \frac{\Delta y(k-1)}{\nu_2} \right|^{p(k-1)} \\
&\quad + \eta \sum_{k=1}^{T} \frac{1}{p(k)} \left| \frac{\nu_1}{\nu_1+\nu_2} \frac{x(k)}{\nu_1} + \left(1 - \frac{\nu_1}{\nu_1+\nu_2}\right) \frac{y(k)}{\nu_2} \right|^{p(k)} \\
&\le \frac{\nu_1}{\nu_1+\nu_2} \mathcal{N}_{x,\eta}(\nu_1) + \frac{\nu_2}{\nu_1+\nu_2} \mathcal{N}_{y,\eta}(\nu_2) \le 1,
\end{aligned}$$

meaning that $\|x+y\|_{\eta,p(\cdot)} \le \nu_1 + \nu_2$ and hence

$$\|x+y\|_{\eta,p(\cdot)} < \|x\|_{\eta,p(\cdot)} + \|y\|_{\eta,p(\cdot)} + 2\varepsilon.$$

Then, letting $\varepsilon \to 0$, one has (2.1.3) and the proof is complete. ∎

It is worth to point out that, if $p =$ constant, then X_{PO} will be endowed with the norm

$$\|x\|_\eta = \left(\sum_{k=1}^{T+1} |\Delta x(k-1)|^p + \eta \sum_{k=1}^{T} |x(k)|^p \right)^{\frac{1}{p}}, \quad (\forall)\ x \in X_{PO},\ (\eta > 0),$$

which simplifies the calculations (see [122]); note that $\|\cdot\|_\eta = p^{\frac{1}{p}} \|\cdot\|_{\eta,p}$.

Also, we shall make use of the usual sup-norm

$$\|x\|_\infty = \max_{k \in \mathbb{Z}[0,T+1]} |x(k)|. \tag{2.1.4}$$

Next, let $\varphi_{X_{PO}} : X_{PO} \to \mathbb{R}$ be defined by

$$\varphi_{X_{PO}}(x) = \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta x(k-1)|^{p(k-1)}, \tag{2.1.5}$$

for all $x \in X_{PO}$. Standard arguments show that $\varphi_{X_{PO}}$ is convex, of class C^1 and its derivative is given by

$$\left\langle \varphi'_{X_{PO}}(x), y \right\rangle = \sum_{k=1}^{T+1} h_{p(k-1)}(\Delta x(k-1)) \Delta y(k-1), \quad (\forall)\ x, y \in X_{PO}. \tag{2.1.6}$$

Clearly, for $x \in X_{PO}$ and $\eta > 0$,

$$\mathcal{N}_{x,\eta}(1) = \varphi_{X_{PO}}(x) + \eta \sum_{k=1}^{T} \frac{|x(k)|^{p(k)}}{p(k)}.$$

The following proposition, which will play a key role in the proofs, establishes the connection between $\varphi_{X_{PO}}$ and the above Luxemburg type norm.

Proposition 2.1.2. *For all $x \in X_{PO}$ and any $\eta > 0$, we have*

$$\|x\|_{\eta,p(\cdot)} > 1 \Rightarrow \|x\|_{\eta,p(\cdot)}^{p^-} \leq \mathcal{N}_{x,\eta}(1) \leq \|x\|_{\eta,p(\cdot)}^{p^+} \tag{2.1.7}$$

and

$$\|x\|_{\eta,p(\cdot)} < 1 \Rightarrow \|x\|_{\eta,p(\cdot)}^{p^+} \leq \mathcal{N}_{x,\eta}(1) \leq \|x\|_{\eta,p(\cdot)}^{p^-}. \tag{2.1.8}$$

Proof. We only prove (2.1.7), because the proof of (2.1.8) is similar. Let $x \in X_{PO}$ and $\eta > 0$. We choose a sequence $\{\nu_n\} \subset (1, \|x\|_{\eta,p(\cdot)})$, so that $\nu_n \to \|x\|_{\eta,p(\cdot)}$. Obviously, for all $n \in \mathbb{N}$, $\nu_n \notin \{\nu > 0 : \mathcal{N}_{x,\eta}(\nu) \leq 1\}$, i.e. $\mathcal{N}_{x,\eta}(\nu_n) > 1$. Since $\nu_n > 1$, we get

$$1 < \mathcal{N}_{x,\eta}(\nu_n) \leq \frac{1}{\nu_n^{p^-}} \mathcal{N}_{x,\eta}(1),$$

for all $n \in \mathbb{N}$. Then, making $n \to \infty$ in $\nu_n^{p^-} < \mathcal{N}_{x,\eta}(1)$, it results

$$\|x\|_{\eta,p(\cdot)}^{p^-} \leq \mathcal{N}_{x,\eta}(1).$$

On the other hand, from the definition of the norm $\|\cdot\|_{\eta,p(\cdot)}$, there exists a sequence $\{\nu_n\} \subset \{\nu > 0 : \mathcal{N}_{x,\eta}(\nu) \leq 1\}$ ($\|x\|_{\eta,p(\cdot)} \leq \nu_n$) with $\nu_n \to \|x\|_{\eta,p(\cdot)}$. Since $\mathcal{N}_{x,\eta}(\nu_n) \leq 1 < \nu_n$, we have

$$\frac{1}{\nu_n^{p^+}} \mathcal{N}_{x,\eta}(1) \leq \mathcal{N}_{x,\eta}(\nu_n) \leq 1,$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in $\mathcal{N}_{x,\eta}(1) \le \nu_n^{p^+}$, it follows

$$\mathcal{N}_{x,\eta}(1) \le \|x\|_{\eta,p(\cdot)}^{p^+},$$

which complete the proof of (2.1.7). ∎

Also, we shall need the following summation by parts formula.

Proposition 2.1.3. *For any $u,\ v : \mathbb{Z} \to \mathbb{R}$, $m \in \mathbb{Z},\ n \in \mathbb{N}$, it holds*

$$\sum_{k=m}^{m+n} \Delta u(k-1)v(k) = u(m+n)v(m+n) - u(m-1)v(m) - \sum_{k=m+1}^{m+n} u(k-1)\Delta v(k-1). \qquad (2.1.9)$$

Proof.

$$\sum_{k=m}^{m+n} \Delta u(k-1)v(k) = \sum_{k=m}^{m+n}(u(k) - u(k-1))v(k)$$
$$= u(m)v(m) - u(m-1)v(m) + u(m+1)v(m+1) - u(m)v(m+1) + \dots$$
$$\dots + u(m+n)v(m+n) - u(m+n-1)v(m+n)$$
$$= u(m+n)v(m+n) - u(m-1)v(m) - u(m)[v(m+1) - v(m)] -$$
$$u(m+1)[v(m+2) - v(m+1)] - u(m+n-1)[v(m+n) - v(m+n-1)]$$
$$= u(m+n)v(m+n) - u(m-1)v(m) - \sum_{k=m+1}^{m+n} u(k-1)\Delta v(k-1).$$

∎

Now, by virtue of (2.1.6), (2.1.9) and (2.1), one has

$$\begin{aligned}
\left\langle \varphi'_{X_{PO}}(x), y \right\rangle &= h_{p(0)}(\Delta x(0))\Delta y(0) + \sum_{k=2}^{T+1} h_{p(k-1)}(\Delta x(k-1))\Delta y(k-1) \\
&= h_{p(0)}(\Delta x(0))y(1) - h_{p(0)}(\Delta x(0))y(0) - h_{p(0)}(\Delta x(0))y(1) \\
&\quad + h_{p(T+1)}(\Delta x(T+1))y(T+1) - \sum_{k=1}^{T+1} \Delta(h_{p(k-1)}(\Delta x(k-1)))y(k) \\
&= -h_{p(0)}(\Delta x(0))y(0) + h_{p(T+1)}(\Delta x(T+1))y(T+1) \\
&\quad - \sum_{k=1}^{T} \Delta(h_{p(k-1)}(\Delta x(k-1)))y(k) + h_{p(T)}(\Delta x(T))y(T+1) \\
&\quad - h_{p(T+1)}(\Delta x(T+1))y(T+1) = h_{p(T)}(\Delta x(T))y(T+1)
\end{aligned}$$

$$-h_{p(0)}(\Delta x(0))y(0) - \sum_{k=1}^{T} \Delta_{p(k-1)}x(k-1)y(k), \tag{2.1.10}$$

for all $x, y \in X_{PO}$.

Next, by means of j, we introduce the functional $J : X_{PO} \to (-\infty, +\infty]$ given by

$$J(x) = j(x(0), x(T+1)), \quad (\forall)\ x \in X_{PO}. \tag{2.1.11}$$

Note that, as j is proper, convex and l.s.c., the same properties hold true for J. Then, with $\varphi_{X_{PO}}$ in (2.1.5), setting

$$\psi = \varphi_{X_{PO}} + J, \tag{2.1.12}$$

it is clear that ψ is proper, convex and l.s.c. on X_{PO}.

Further, denoting by $F : \mathbb{Z}[1, T] \times \mathbb{R} \to \mathbb{R}$ the primitive of f, i.e.,

$$F(k, t) = \int_0^t f(k, \tau)d\tau, \quad (\forall)\ k \in \mathbb{Z}[1, T],\ t \in \mathbb{R}, \tag{2.1.13}$$

we define

$$\mathcal{F}_{X_{PO}}(x) = -\sum_{k=1}^{T} F(k, x(k)), \quad (\forall)\ x \in X_{PO}. \tag{2.1.14}$$

It is not difficult to check that $\mathcal{F}_{X_{PO}} \in C^1(X_{PO}, \mathbb{R})$ and

$$\left\langle \mathcal{F}'_{X_{PO}}(x), y \right\rangle = -\sum_{k=1}^{T} f(k, x(k))y(k), \quad (\forall)\ x, y \in X_{PO}. \tag{2.1.15}$$

The energy functional associated to problem (2.1.1), (2.1.2) is

$$\mathcal{I} = \mathcal{F}_{X_{PO}} + \psi,$$

with ψ in (2.1.12) and $\mathcal{F}_{X_{PO}}$ given by (2.1.14).

Proposition 2.1.4. *If $x \in X_{PO}$ is a critical point of the functional $\mathcal{I}$ in the sense that (see* (1.12)*)*

$$\left\langle \mathcal{F}'_{X_{PO}}(x), y - x \right\rangle + \psi(y) - \psi(x) \geq 0, \quad (\forall)\ y \in X_{PO}, \tag{2.1.16}$$

then x is a solution of problem (2.1.1)*,* (2.1.2)*.*

Proof. Assume that $x \in X_{PO}$ is a critical point of $\mathcal{I}$. In (2.1.16) we take $y = x + sw,\ s > 0$; then dividing by s and letting $s \to 0^+$, we obtain

$$\left\langle \mathcal{F}'_{X_{PO}}(x), w \right\rangle + \left\langle \varphi'_{X_{PO}}(x), w \right\rangle + J'(x; w) \geq 0, \quad (\forall)\ w \in X_{PO},$$

where $J'(x;w)$ is the directional derivative of the convex function J at x in the direction of w; this is known to exist. By the definition of J (see (2.1.11)), the above inequality becomes

$$\left\langle \mathcal{F}'_{X_{PO}}(x), w \right\rangle + \left\langle \varphi'_{X_{PO}}(x), w \right\rangle + j'((x(0), x(T+1)); (w(0), w(T+1))) \geq 0,$$

for all $w \in X_{PO}$. From (2.1.15) and (2.1.10), one has

$$-\sum_{k=1}^{T} f(k, x(k))w(k) + h_{p(T)}(\Delta x(T))w(T+1) - h_{p(0)}(\Delta x(0))w(0)$$
$$-\sum_{k=1}^{T} \Delta_{p(k-1)} x(k-1)w(k) + j'((x(0), x(T+1)); (w(0), w(T+1))) \geq 0, \tag{2.1.17}$$

for all $w \in X_{PO}$. Thus, we infer

$$\sum_{k=1}^{T} (\Delta_{p(k-1)} x(k-1) + f(k, x(k)))w(k) = 0,$$

for all $w \in X_{PO}$ with $w(0) = w(T+1) = 0$. This implies that

$$-\Delta_{p(k-1)} x(k-1) = f(k, x(k)), \quad (\forall)\ k \in \mathbb{Z}[1, T]. \tag{2.1.18}$$

To prove that x satisfies condition (2.1.2), we multiply the equality (2.1.18) by $w(k)$. Then summing from 1 to T and using (2.1.17), one obtains

$$\begin{aligned} &j'((x(0),\ x(T+1)); (w(0), w(T+1))) \geq \\ &\qquad - h_{p(T)}(\Delta x(T))w(T+1) + h_{p(0)}(\Delta x(0))w(0), \end{aligned}$$

for all $w \in X_{PO}$. Taking $w \in X_{PO}$ with $w(0) = u$ and $w(T+1) = v$, where $u, v \in \mathbb{R}$ are arbitrarily chosen, we have

$$j'((x(0), x(T+1)); (u, v)) \geq h_{p(0)}(\Delta x(0))u - h_{p(T)}(\Delta x(T))v, \quad (\forall)\ u, v \in \mathbb{R},$$

which, by a standard result from convex analysis (see, e.g., Theorem 23.2 in [119]), means that

$$(h_{p(0)}(\Delta x(0)), -h_{p(T)}(\Delta x(T))) \in \partial j(x(0), x(T+1))$$

and the proof is complete. ∎

2.1.2 Ground state solutions

We begin by introducing the following constant

$$\lambda_1 := \inf \left\{ \frac{\sum_{k=1}^{T+1} \frac{|\Delta x(k-1)|^{p(k-1)}}{p(k-1)}}{\sum_{k=1}^{T} \frac{|x(k)|^{p(k)}}{p(k)}} : x \in X_{PO} \setminus \{0\}, (x(0), x(T+1)) \in D(j) \right\}. \tag{2.1.19}$$

The main result here states that if the potential of the nonlinearity f lies asymptotically on the left of λ_1, then problem (2.1.1), (2.1.2) is solvable.

Theorem 2.1.5. *If*

$$\limsup_{|t|\to\infty} \frac{p(k)F(k,t)}{|t|^{p(k)}} < \lambda_1, \quad (\forall)\ k \in \mathbb{Z}[1,T], \tag{2.1.20}$$

then problem (2.1.1), (2.1.2) *has at least one solution which minimizes the energy functional $\mathcal{I}$ on X_{PO}.*

Proof. We shall prove that $\mathcal{I}$ is coercive on X_{PO}. Then, by the continuity of $\mathcal{F}_{X_{PO}}$, the lower semicontinuity of ψ and the direct method in calculus of variations (see Theorem 1.2), $\mathcal{I}$ is bounded from below and attains its infimum at some $x \in X_{PO}$, which on account of Proposition 1.12 and Proposition 2.1.4, is a solution of problem (2.1.1), (2.1.2).

From (2.1.20) there are constants $\sigma > 0$ and $\rho > 0$ such that

$$F(k,t) \le \frac{\lambda_1 - \sigma}{p(k)} |t|^{p(k)}, \quad (\forall)\ k \in \mathbb{Z}[1,T],\ (\forall)\ t \in \mathbb{R} \text{ with } |t| > \rho.$$

If $\lambda_1 > 0$, we may assume that $\sigma < \lambda_1$. On the other hand, by the continuity of F, there is a constant $M_\rho > 0$ such that

$$|F(k,t)| \le M_\rho, \quad (\forall)\ k \in \mathbb{Z}[1,T],\ (\forall)\ t \in \mathbb{R} \text{ with } |t| \le \rho.$$

Hence, we infer

$$F(k,t) \le M_\rho + \frac{|\lambda_1 - \sigma|}{p(k)} \rho^{p(k)} + \frac{\lambda_1 - \sigma}{p(k)} |t|^{p(k)}, \quad (\forall)\ k \in \mathbb{Z}[1,T],\ (\forall)\ t \in \mathbb{R}.$$

Using this inequality, we estimate $\mathcal{I}$ as follows

$$\mathcal{I}(x) \ge \varphi_{X_{PO}}(x) - M_\rho T - |\lambda_1 - \sigma| \sum_{k=1}^{T} \frac{\rho^{p(k)}}{p(k)} - (\lambda_1 - \sigma) \sum_{k=1}^{T} \frac{|x(k)|^{p(k)}}{p(k)} + J(x)$$

$$\ge \varphi_{X_{PO}}(x) - M_\rho T - \frac{|\lambda_1 - \sigma|}{p^-} (\rho^{p^-} + \rho^{p^+}) T - (\lambda_1 - \sigma) \sum_{k=1}^{T} \frac{|x(k)|^{p(k)}}{p(k)} + J(x)$$

$$= \varphi_{X_{PO}}(x) - (\lambda_1 - \sigma) \sum_{k=1}^{T} \frac{|x(k)|^{p(k)}}{p(k)} - C_1 + J(x), \qquad (\forall)\ x \in X_{PO},$$

where $C_1 = M_\rho T + \dfrac{|\lambda_1 - \sigma|}{p^-}(\rho^{p^-} + \rho^{p^+})T.$

If $\lambda_1 = 0$, using (2.1.7) from Proposition 2.1.2, we have

$$\mathcal{I}(x) \geq \varphi_{X_{PO}}(x) + \sigma \sum_{k=1}^{T} \frac{|x(k)|^{p(k)}}{p(k)} - C_1 + J(x) = \mathcal{N}_{x,\sigma}(1) - C_1 + J(x)$$
$$\geq \|x\|_{\sigma,p(\cdot)}^{p^-} - C_1 + J(x), \quad (\forall)\ x \in X_{PO},\ \|x\|_{\sigma,p(\cdot)} > 1. \qquad (2.1.21)$$

In the case $\lambda_1 > 0$, by virtue of (2.1.7) and (2.1.19), for $\|x\|_{\lambda_1,p(\cdot)} > 1$, one obtains

$$\|x\|_{\lambda_1,p(\cdot)}^{p^-} \leq \mathcal{N}_{x,\lambda_1}(1) = \varphi_{X_{PO}}(x) + \lambda_1 \sum_{k=1}^{T} \frac{|x(k)|^{p(k)}}{p(k)} \leq 2\varphi_{X_{PO}}(x),$$

which, using again (2.1.19), implies

$$\mathcal{I}(x) \geq \varphi_{X_{PO}}(x) + (\sigma - \lambda_1) \frac{\sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta x(k-1)|^{p(k-1)}}{\lambda_1} - C_1 + J(x)$$
$$= \frac{\sigma}{\lambda_1} \varphi_{X_{PO}}(x) - C_1 + J(x) \geq \frac{\sigma}{2\lambda_1} \|x\|_{\lambda_1,p(\cdot)}^{p^-} - C_1 + J(x), \qquad (2.1.22)$$

for all $x \in D(J)$. In both cases, by virtue of (2.1.21) and (2.1.22), there exist constants $\eta,\ C_2 > 0$ such that

$$\mathcal{I}(x) \geq C_2 \|x\|_{\eta,p(\cdot)}^{p^-} - C_1 + J(x), \quad (\forall)\ x \in D(J),\ \|x\|_{\eta,p(\cdot)} > 1.$$

On the other hand, as j is convex, proper and l.s.c., it is bounded from below by an affine functional (see Proposition 1.3). Therefore, on account of (2.1.11), there are positive constants k_1, k_2, k_3 such that

$$\mathcal{I}(x) \geq C_2 \|x\|_{\eta,p(\cdot)}^{p^-} - C_1 - k_1|x(0)| - k_2|x(T+1)| - k_3$$
$$\geq C_2 \|x\|_{\eta,p(\cdot)}^{p^-} - C_3\|x\|_\infty - C_4, \quad (\forall)\ x \in D(J),$$

with $C_3 = k_1 + k_2$ and $C_4 = C_1 + k_3$. Since any norm on X_{PO} is equivalent with $\|\cdot\|_{\eta,p(\cdot)}$, there exists $C_5 > 0$ so that

$$\mathcal{I}(x) \geq C_2 \|x\|_{\eta,p(\cdot)}^{p^-} - C_5\|x\|_{\eta,p(\cdot)} - C_4.$$

Consequently,

$$\mathcal{I}(x) \to +\infty, \text{ as } \|x\|_{\eta,p(\cdot)} \to \infty,$$

meaning that $\mathcal{I}$ is coercive on $(X_{PO}, \|\cdot\|_{\eta,p(\cdot)})$ and the proof is complete. ■

Remark 2.1.6. Theorem 3.1 proved in [122] for $p =$ constant is a simple consequence of Theorem 2.1.5.

Example 2.1.7 (see Example 3.2 in [122] for $p =$ constant)**.** If $j : \mathbb{R}\times\mathbb{R} \to (-\infty, +\infty]$ is proper, convex and l.s.c., then problem

$$\begin{cases} -\Delta_{p(k-1)}x(k-1) + h_{p(k)}(x(k)) = \ell(k), \quad (\forall)\ k \in \mathbb{Z}[1,T], \\ \\ (h_{p(0)}(\Delta x(0)), -h_{p(T)}(\Delta x(T))) \in \partial j(x(0), x(T+1)) \end{cases}$$

is solvable for any $\ell \in \mathbb{R}^{\mathbb{Z}[1,T]}$. Indeed, here,

$$F(k,t) = \ell(k)t - \frac{|t|^{p(k)}}{p(k)}$$

and so,

$$\lim_{|t|\to\infty} \frac{p(k)F(k,t)}{|t|^{p(k)}} = -1 < \lambda_1, \quad (\forall)\ k \in \mathbb{Z}[1,T].$$

Now, we consider the problem

$$\begin{cases} -\Delta_{p(k-1)}x(k-1) = \lambda g(k, x(k)), \quad (\forall)\ k \in \mathbb{Z}[1,T], \\ \\ (h_{p(0)}(\Delta x(0)), -h_{p(T)}(\Delta x(T))) \in \partial j(x(0), x(T+1)), \end{cases} \tag{2.1.23}$$

where $g : \mathbb{Z}[1,T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and λ is a positive parameter.

Corollary 2.1.8. *Assume that $\lambda_1 > 0$ and g satisfies the growth condition*

$$|g(k,t)| \le a|t|^{q(k)} + b, \qquad (\forall)\ k \in \mathbb{Z}[1,T],\ (\forall)\ t \in \mathbb{R}, \tag{2.1.24}$$

where $a > 0$, $b \in \mathbb{R}_+$ are constants and $q : \mathbb{Z}[1,T] \to [0,\infty)$. The following hold true:

(i) if $\underline{p} > \overline{q} + 1$, then problem (2.1.23) has a solution for any $\lambda > 0$;

(ii) if $\underline{p} = \overline{q} + 1$, then there is some $\lambda^\star > 0$ such that for any $\lambda \in (0, \lambda^\star)$, problem (2.1.23) is solvable.

Proof. We apply Theorem 2.1.5 with $f(k,t) = \lambda g(k,t)$, for all $k \in \mathbb{Z}[1,T]$ and $t \in \mathbb{R}$. From (2.1.24), we obtain

$$|F(k,t)| = \left|\int_0^t f(k,\tau)d\tau\right| \le \lambda \int_0^{|t|} \left(a\tau^{q(k)} + b\right) d\tau$$
$$= \lambda a \frac{|t|^{q(k)+1}}{q(k)+1} + \lambda|t|b \le \frac{\lambda a}{\underline{q}+1}|t|^{q(k)+1} + \lambda|t|b,$$

for all $k \in \mathbb{Z}[1,T]$ and all $t \in \mathbb{R}$. Thus, we deduce

$$\frac{p(k)F(k,t)}{|t|^{p(k)}} \le \frac{\overline{p}\lambda a}{\underline{q}+1}\frac{|t|^{q(k)+1}}{|t|^{p(k)}} + \frac{\overline{p}\lambda b|t|}{|t|^{p(k)}} \le \frac{\overline{p}\lambda a}{\underline{q}+1}\frac{|t|^{\overline{q}+1}}{|t|^{\underline{p}}} + \frac{\overline{p}\lambda b|t|}{|t|^{\underline{p}}},$$

for all $k \in \mathbb{Z}[1,T]$ and $t \in \mathbb{R}$ with $|t| > 1$. So, if $\underline{p} > \overline{q} + 1$ then

$$\limsup_{|t|\to\infty} \frac{p(k)F(k,t)}{|t|^{p(k)}} \le 0 < \lambda_1, \quad (\forall)\ k \in \mathbb{Z}[1,T].$$

Also, if $\underline{p} = \overline{q} + 1$, setting

$$\lambda^\star = \frac{\lambda_1(\underline{q}+1)}{a\overline{p}}, \tag{2.1.25}$$

easy it can be seen that condition (2.1.20) is fulfilled for any $\lambda \in (0, \lambda^\star)$. ■

Remark 2.1.9. (i) Note that a valid $\lambda^\star$ in Corollary 2.1.8 (ii) is given by formula (2.1.25).

(ii) If $\lambda_1 > 0$, then an immediate consequence of Corollary 2.1.8 is Theorem 5 from M. Galewski and S. Głąb [67] – one choose $j =$ the indicator function of the set $K = \{(0,0)\}$.

2.1.3 Mountain pass type solutions

Let $r : \mathbb{Z}[1,T] \to [0,\infty)$ be a given function. In this paragraph we study the existence of nontrivial solutions for the equation

$$-\Delta_{p(k-1)}x(k-1) + r(k)h_{p(k)}(x(k)) = f(k,x(k)), \quad (\forall)\ k \in \mathbb{Z}[1,T], \tag{2.1.26}$$

associated with the potential boundary condition (2.1.2). The main tool in obtaining such a result will be the Mountain Pass Theorem for Szulkin's type functionals (Theorem 1.14).

To treat problem (2.1.26), (2.1.2), instead of $\varphi_{X_{PO}}$ we shall consider $\Phi_{X_{PO}} : X_{PO} \to \mathbb{R}$ defined by

$$\Phi_{X_{PO}}(x) = \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta x(k-1)|^{p(k-1)} + \sum_{k=1}^{T} \frac{r(k)}{p(k)} |x(k)|^{p(k)}, \quad (\forall)\ x \in X_{PO}, \tag{2.1.27}$$

which is of class C^1 on X_{PO} and its derivative is given by

$$\left\langle \Phi'_{X_{PO}}(x), y \right\rangle = \sum_{k=1}^{T+1} h_{p(k-1)}(\Delta x(k-1)) \Delta y(k-1) + \sum_{k=1}^{T} r(k) h_{p(k)}(x(k)) y(k), \tag{2.1.28}$$

for all $x,\ y \in X_{PO}$. Then, setting $\psi_r = \Phi_{X_{PO}} + J$, we define

$$\mathcal{I}_r = \mathcal{F}_{X_{PO}} + \psi_r, \tag{2.1.29}$$

with J given by (2.1.11) and $\mathcal{F}_{X_{PO}}$ in (2.1.14).

By means of λ_1 in (2.1.19) we define the constants

$$\underline{\eta} := \lambda_1 + \underline{r} \ \text{ and } \ \overline{\eta} := \lambda_1 + \overline{r}. \tag{2.1.30}$$

The following lemma provides sufficient conditions for $\mathcal{I}_r$ given in (2.1.29) to satisfy the (PS) condition in the sense of Szulkin.

Lemma 2.1.10. *If $\underline{\eta} > 0$ and there exist constants $\theta > p^+$ and $C_1, \rho > 0$ such that*

$$j'(z; z) \leq \theta j(z) + C_1, \quad (\forall)\ z \in D(j) \tag{2.1.31}$$

and

$$\theta F(k,t) \leq t f(k,t), \quad (\forall)\ k \in \mathbb{Z}[1,T],\ (\forall)\ t \in \mathbb{R} \text{ with } |t| > \rho, \tag{2.1.32}$$

then the functional $\mathcal{I}_r$ satisfies the (PS) condition on $(X_{PO}, \|\cdot\|_{\underline{\eta},p(\cdot)})$ in the sense that (see Definition 1.13), every sequence $\{x_n\} \subset X_{PO}$ for which $\mathcal{I}_r(x_n) \to c \in \mathbb{R}$ and

$$\left\langle \mathcal{F}'_{X_{PO}}(x_n), y - x_n \right\rangle + \psi_r(y) - \psi_r(x_n) \geq -\varepsilon_n \|y - x_n\|_{\underline{\eta},p(\cdot)}, \quad (\forall)\ y \in X_{PO}, \tag{2.1.33}$$

where $\varepsilon_n \to 0$, possesses a convergent subsequence.

Proof. Let $\{x_n\} \subset X_{PO}$ be a sequence for which $\mathcal{I}_r(x_n) \to c \in \mathbb{R}$ and (2.1.33) holds true with $\varepsilon_n \to 0$. It suffices to show that $\{x_n\}$ is bounded. We may assume that $\{x_n\} \subset D(\mathcal{I}_r) = D(J)$ and $\|x_n\|_{\underline{\eta},p(\cdot)} > 1$ for all $n \in \mathbb{N}$. From (2.1.11) and (2.1.31) it follows

$$J(y) - \frac{1}{\theta} J'(y; y) \geq -C_2, \quad (\forall)\ y \in D(J), \tag{2.1.34}$$

with $C_2 = C_1/\theta$. Using (2.1.32) we deduce that, for all $n \in \mathbb{N}$, it holds

$$\begin{aligned}\frac{1}{\theta}\left\langle \mathcal{F}'_{X_{PO}}(x_n), x_n\right\rangle - \mathcal{F}_{X_{PO}}(x_n) &= \frac{1}{\theta}\sum_{k=1}^{T}[\theta F(k, x_n(k)) - x_n(k)f(k, x_n(k))] \\ &\le \frac{1}{\theta}\sum_{|x_n(k)|\le\rho}[\theta F(k, x_n(k)) - x_n(k)f(k, x_n(k))] \\ &\le \frac{1}{\theta}\sum_{k=1}^{T}\max_{|x|\le\rho}|\theta F(k, x) - xf(k, x)| =: C_3. \qquad (2.1.35)\end{aligned}$$

Clearly, there is a constant $C_4 > 0$, such that

$$|\mathcal{I}_r(x_n)| \le C_4, \quad (\forall)\ n \in \mathbb{N}. \qquad (2.1.36)$$

Now, setting $y = x_n + sx_n$ in (2.1.33), dividing by $s > 0$ and letting $s \to 0^+$, we obtain

$$\left\langle \mathcal{F}'_{X_{PO}}(x_n), x_n\right\rangle + \left\langle \Phi'_{X_{PO}}(x_n), x_n\right\rangle + J'(x_n; x_n) \ge -\varepsilon_n\|x_n\|_{\underline{\eta},p(\cdot)},$$

for all $n \in \mathbb{N}$. This, together with (2.1.36), implies

$$\begin{aligned}C_4 + \frac{\varepsilon_n}{\theta}\|x_n\|_{\underline{\eta},p(\cdot)} &\ge \mathcal{F}_{X_{PO}}(x_n) + \Phi_{X_{PO}}(x_n) + J(x_n) + \frac{\varepsilon_n}{\theta}\|x_n\|_{\underline{\eta},p(\cdot)} \\ &\ge \mathcal{F}_{X_{PO}}(x_n) - \frac{1}{\theta}\left\langle \mathcal{F}'_{X_{PO}}(x_n), x_n\right\rangle + \Phi_{X_{PO}}(x_n) \\ &\quad - \frac{1}{\theta}\left\langle \Phi'_{X_{PO}}(x_n), x_n\right\rangle + J(x_n) - \frac{1}{\theta}J'(x_n; x_n),\end{aligned}$$

and by virtue of (2.1.34), (2.1.35), (2.1.27) and (2.1.28), we have

$$C_2 + C_3 + C_4 + \frac{\varepsilon_n}{\theta}\|x_n\|_{\underline{\eta},p(\cdot)} \ge$$

$$\left(\frac{1}{p^+} - \frac{1}{\theta}\right)\left(\sum_{k=1}^{T+1}|\Delta x_n(k-1)|^{p(k-1)} + \sum_{k=1}^{T}r(k)|x_n(k)|^{p(k)}\right). \qquad (2.1.37)$$

Further, on account of (2.1.7), (2.1.30) and (2.1.19), we deduce

$$\begin{aligned}\|x_n\|_{\underline{\eta},p(\cdot)}^{p^-} \le \mathcal{N}_{x_n,\underline{\eta}}(1) &= \sum_{k=1}^{T+1}\frac{|\Delta x_n(k-1)|^{p(k-1)}}{p(k-1)} + (\lambda_1 + \underline{r})\sum_{k=1}^{T}\frac{|x_n(k)|^{p(k)}}{p(k)} \\ &\le 2\left(\sum_{k=1}^{T+1}\frac{|\Delta x_n(k-1)|^{p(k-1)}}{p(k-1)} + \sum_{k=1}^{T}\frac{r(k)}{p(k)}|x_n(k)|^{p(k)}\right) \\ &\le \frac{2}{p^-}\left(\sum_{k=1}^{T+1}|\Delta x_n(k-1)|^{p(k-1)} + \sum_{k=1}^{T}r(k)|x_n(k)|^{p(k)}\right).\end{aligned}$$

Using the above inequality in (2.1.37), one has

$$C_2 + C_3 + C_4 + \frac{\varepsilon_n}{\theta}\|x_n\|_{\underline{\eta},p(\cdot)} \geq \frac{p^-}{2}\Big(\frac{1}{p^+} - \frac{1}{\theta}\Big)\|x_n\|^{p^-}_{\underline{\eta},p(\cdot)}.$$

Since $\theta > p^+$, we infer that $\{x_n\}$ is bounded, which complete the proof. ■

Proposition 2.1.11. *If there are constants $\theta > 0$ and $\rho > 0$ such that the Ambrosetti-Rabinowitz type condition [5]*

$$0 < \theta F(k,t) \leq t f(k,t), \quad (\forall)\ k \in \mathbb{Z}[1,T],\ (\forall)\ t \in \mathbb{R} \text{ with } |t| > \rho, \tag{2.1.38}$$

holds true, then there exist $a_1, a_2 > 0$ such that

$$F(k,t) \geq a_1|t|^\theta - a_2, \quad (\forall)\ k \in \mathbb{Z}[1,T],\ (\forall)\ t \in \mathbb{R}.$$

Proof. Let $t \in \mathbb{R}$ with $t > \rho$. By (2.1.38), for all $k \in \mathbb{Z}[1,T]$ and $\tau \in (\rho, t)$, we have

$$\frac{\theta}{\tau} \leq \frac{f(k,\tau)}{F(k,\tau)} = \frac{F'_\tau(k,\tau)}{F(k,\tau)}$$

and integrating from ρ to t, one gets

$$\ln\left(\frac{t}{\rho}\right)^\theta \leq \ln F(k,t) - \ln F(k,\rho) \ \Leftrightarrow\ \ln F(k,t) \geq \ln\frac{t^\theta F(k,\rho)}{\rho^\theta},$$

i.e.,

$$F(k,t) \geq \frac{F(k,\rho)}{\rho^\theta} t^\theta \geq \frac{a_{11}}{\rho^\theta}|t|^\theta,$$

where $a_{11} = \min_{k\in\mathbb{Z}[1,T]} F(k,\rho)$. For all $t \in \mathbb{R}$, $t < -\rho$, using the same arguments, we deduce that

$$F(k,t) \geq \frac{F(k,-\rho)}{\rho^\theta}(-t)^\theta \geq \frac{a_{12}}{\rho^\theta}|t|^\theta,$$

where $a_{12} = \min_{k\in\mathbb{Z}[1,T]} F(k,-\rho)$. Hence, with $a_1 := \min\{a_{11}, a_{12}\}/\rho^\theta > 0$, we have that

$$F(k,t) \geq a_1|t|^\theta, \quad (\forall)\ k \in \mathbb{Z}[1,T],\ (\forall)\ t \in \mathbb{R} \text{ with } |t| > \rho.$$

On the other hand, by the continuity of F, there is a constant $M > 0$ such that

$$|F(k,t)| \leq M, \quad (\forall)\ k \in \mathbb{Z}[1,T],\ (\forall)\ t \in \mathbb{R} \text{ with } |t| \leq \rho.$$

Setting $a_2 := M + a_1\rho^\theta$, it follows

$$F(k,t) \geq a_1|t|^\theta - a_2, \quad (\forall)\ k \in \mathbb{Z}[1,T],\ (\forall)\ t \in \mathbb{R},$$

as claimed. ∎

Now, we can state the following main result.

Theorem 2.1.12. *Assume that* $\underline{\eta} > 0$ *and, in addition,*

(i) $(0,0) \in \partial j(0,0)$;

(ii) $\limsup_{|t|\to 0} \dfrac{p(k)F(k,t)}{|t|^{p(k)}} < \underline{\eta}, \quad (\forall)\ k \in \mathbb{Z}[1,T]$;

(iii) there are constants $\theta > p^+$ *and* $C_1,\ \rho > 0$ *such that* (2.1.31) *and* (2.1.38) *hold true.*

Then, problem (2.1.26)*,* (2.1.2) *has a nontrivial solution.*

Proof. Without loss of generality, we may assume that

$$j(0,0) = 0, \tag{2.1.39}$$

which implies that $\mathcal{I}_r(0) = 0$. From (i), (2.1.11) and (2.1.39), we have

$$J(x) \geq J(0) = 0, \quad (\forall)\ x \in D(J). \tag{2.1.40}$$

On account of Lemma 2.1.10, it is clear that condition (iii) ensures that the functional $\mathcal{I}_r$ satisfies the (PS) condition on $(X_{PO}, \|\cdot\|_{\underline{\eta},p(\cdot)})$. It remains to show that $\mathcal{I}_r$ has the geometry required by Theorem 1.14.

By the equivalence of the norms on X_{PO}, there is some $C_2 > 0$ so that

$$\|x\|_\infty \leq C_2\|x\|_{\underline{\eta},p(\cdot)}, \quad (\forall)\ x \in X_{PO}. \tag{2.1.41}$$

Using (ii) we can find constants $\sigma \in (0,\underline{\eta})$ and $\mu \in (0,1)$ such that

$$F(k,t) \leq \frac{\underline{\eta}-\sigma}{p(k)}|t|^{p(k)}, \quad (\forall)\ k \in \mathbb{Z}[1,T],\ (\forall)\ t \in \mathbb{R} \text{ with } |t| \leq \mu C_2. \tag{2.1.42}$$

Let $x \in X_{PO}$, with $\|x\|_{\underline{\eta},p(\cdot)} \leq \mu$, be arbitrarily chosen. From (2.1.41) and (2.1.42), we have

$$F(k,x(k)) \leq \frac{\underline{\eta}-\sigma}{p(k)}|x(k)|^{p(k)}, \quad (\forall)\ k \in \mathbb{Z}[1,T],$$

which implies

$$-\mathcal{F}_{X_{PO}}(x) \leq (\underline{\eta} - \sigma) \sum_{k=1}^{T} \frac{1}{p(k)} |x(k)|^{p(k)}.$$

Further, using (2.1.8) and (2.1.19), easily follows that

$$\|x\|_{\underline{\eta}, p(\cdot)}^{p^+} \leq \mathcal{N}_{x,\underline{\eta}}(1) \leq 2\Phi_{X_{PO}}(x). \tag{2.1.43}$$

By virtue of (2.1.19), (2.1.30), (2.1.27) and (2.1.43), we deduce

$$\mathcal{F}_{X_{PO}}(x) + \Phi_{X_{PO}}(x) \geq \Phi_{X_{PO}}(x) + (\sigma - \underline{\eta}) \sum_{k=1}^{T} \frac{1}{p(k)} |x(k)|^{p(k)}$$

$$\geq \Phi_{X_{PO}}(x) + (\sigma - \underline{\eta}) \frac{\sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta x(k-1)|^{p(k-1)} + \sum_{k=1}^{T} \frac{r(k)}{p(k)} |x(k)|^{p(k)}}{\underline{\eta}}$$

$$= \frac{\sigma}{\underline{\eta}} \Phi_{X_{PO}}(x) \geq \frac{\sigma}{2\underline{\eta}} \|x\|_{\underline{\eta}, p(\cdot)}^{p^+}, \quad (\forall)\ x \in D(J), \quad \|x\|_{\underline{\eta}, p(\cdot)} \leq \mu.$$

Then, on account of (2.1.40), we infer that

$$\mathcal{I}_r(x) = \mathcal{F}_{X_{PO}}(x) + \Phi_{X_{PO}}(x) + J(x) \geq \alpha, \quad \text{if } \|x\|_{\underline{\eta}, p(\cdot)} = \mu, \tag{2.1.44}$$

with $(\sigma/2\underline{\eta})\mu^{p^+} =: \alpha > 0$ and so condition (i) in Theorem 1.14 is fulfilled.

Our next task is to prove that $\mathcal{I}_r$ satisfies Theorem 1.14 (ii). To this end, let $x_0 \in X_{PO} \setminus \{0\}$ be such that $x_0(0) = x_0(T+1) = 0$ and $\|x_0\|_{\overline{\eta}, p(\cdot)} > 1$. Using (2.1.7), (2.1.30) and (2.1.27), one obtains

$$\begin{aligned} \|x_0\|_{\overline{\eta}, p(\cdot)}^{p^+} \geq \mathcal{N}_{x_0, \overline{\eta}}(1) &= \sum_{k=1}^{T+1} \frac{|\Delta x_0(k-1)|^{p(k-1)}}{p(k-1)} + (\lambda_1 + \overline{r}) \sum_{k=1}^{T} \frac{|x_0(k)|^{p(k)}}{p(k)} \\ &\geq \sum_{k=1}^{T+1} \frac{|\Delta x_0(k-1)|^{p(k-1)}}{p(k-1)} + \overline{r} \sum_{k=1}^{T} \frac{|x_0(k)|^{p(k)}}{p(k)} \geq \Phi_{X_{PO}}(x_0). \end{aligned} \tag{2.1.45}$$

From (2.1.39), we have that

$$J(sx_0) = 0, \quad (\forall)\ s \in \mathbb{R},$$

which, together with (2.1.45) and Proposition 2.1.11, for any $s \geq 1$, gives

$$\begin{aligned} \mathcal{I}_r(sx_0) &\leq s^{p^+} \Phi_{X_{PO}}(x_0) + \mathcal{F}_{X_{PO}}(sx_0) \\ &\leq s^{p^+} \|x_0\|_{\overline{\eta}, p(\cdot)}^{p^+} - s^{\theta} a_1 \sum_{k=1}^{T} |x_0(k)|^{\theta} + a_2 T \to -\infty, \end{aligned}$$

as $s \to +\infty$, because $\theta > p^+$. Hence, we can choose s_0 large enough to satisfy $\mathcal{I}_r(s_0x_0) \leq 0$ and $\|s_0x_0\|_{\underline{\eta},p(\cdot)} > \mu$, with μ entering in (2.1.44). This means that condition (ii) in Theorem 1.14 is satisfied with $e = s_0x_0$.

Consequently, the energy functional $\mathcal{I}_r$ has a nontrivial critical point $x \in X_{PO}$, which on account of Proposition 2.1.4 is a nontrivial solution of problem (2.1.26), (2.1.2). ∎

Remark 2.1.13. (i) Theorem 2.1.12 recovers Theorem 4.4 proved in [122] for the case when p is constant.

(ii) Assumptions (i) and (ii) in Theorem 2.1.12 ensure that problem (2.1.26), (2.1.2) also admits the trivial solution (see [76, Remark 3.13 (ii)]).

(iii) According to Remark 3.13 (i) in [76], under the hypothesis (i) of Theorem 2.1.12 and (2.1.31), $D(j)$ necessarily is a cone if it is closed.

We conclude by an application of Theorem 2.1.12.

Let $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a convex and Gâteaux differentiable function, with $dg(0,0) = (0,0)$, where dg denotes the differential of g. Also, let $K \subset \mathbb{R}\times\mathbb{R}$ be a nonempty closed convex cone. We denote by $N_K(z)$ the normal cone to K at $z \in K$, i.e.,

$$N_K(z) = \{\zeta \in \mathbb{R}\times\mathbb{R} : \ (\zeta|\xi - z) \leq 0, \ (\forall)\ \xi \in K\}, \quad (\forall)\ z \in K.$$

The equation (2.1.26) is considered to be associated with the boundary conditions

$$\begin{aligned} &(x(0), x(T+1)) \in K, \\ &(h_{p(0)}(\Delta x(0)), -h_{p(T)}(\Delta x(T))) - dg(x(0), x(T+1)) \in N_K(x(0), x(T+1)). \end{aligned} \tag{2.1.46}$$

We set

$$\underline{\eta}_K := \underline{r} + \inf\left\{ \frac{\sum_{k=1}^{T+1} \frac{|\Delta x(k-1)|^{p(k-1)}}{p(k-1)}}{\sum_{k=1}^{T} \frac{|x(k)|^{p(k)}}{p(k)}} : x \in X_{PO}\setminus\{0\}, (x(0), x(T+1)) \in K \right\}.$$

Theorem 2.1.14. *If $f : \mathbb{Z}[1,T]\times\mathbb{R} \to \mathbb{R}$ is continuous, $\underline{\eta}_K > 0$ and in addition, we assume that*

(i) $\displaystyle \limsup_{|t|\to 0} \frac{p(k)F(k,t)}{|t|^{p(k)}} < \underline{\eta}_K, \quad (\forall)\ k \in \mathbb{Z}[1,T];$

(ii) there are constants $\theta > p^+$ and $C_1,\ \rho > 0$ such that (2.1.38) *holds true and*

$$\langle dg(z), z\rangle \leq \theta g(z) + C_1, \quad (\forall)\ z \in K, \tag{2.1.47}$$

then problem (2.1.26), (2.1.46) *has a nontrivial solution.*

Proof. Let I_K be the indicator function of the cone K. Since

$$N_K(z) = \partial I_K(z), \quad (\forall)\ z \in K,$$

Theorem 2.1.12 applies with $j(z) = g(z) + I_K(z)$ for all $z \in \mathbb{R} \times \mathbb{R}$. ■

Remark 2.1.15. Conditions (2.1.46) allow various possible choices of g and K, which, among others, recover classical boundary conditions. For instance, if $g = 0$, then the homogeneous boundary conditions

$$x(0) = 0 = x(T+1), \quad (\textit{Dirichlet})$$

$$\Delta x(0) = 0 = \Delta x(T), \quad (\textit{Neumann})$$

are obtained by choosing $K = \{(0,0)\}$, respectively $K = \mathbb{R} \times \mathbb{R}$.

If, in addition, $p(0) = p(T)$, then taking $K = \{(x,x),\ x \in \mathbb{R}\}$ and $K = \{(x,-x),\ x \in \mathbb{R}\}$, we get

$$x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T), \quad (\textit{periodic})$$

$$x(0) + x(T+1) = 0 = \Delta x(0) + \Delta x(T), \quad (\textit{antiperiodic})$$

respectively. If the T-periodicity condition over p is not assumed, then we only have

$$h_{p(0)}(\Delta x(0)) = h_{p(T)}(\Delta x(T)) \ \text{ and } \ h_{p(0)}(\Delta x(0)) = -h_{p(T)}(\Delta x(T)),$$

instead of $\Delta x(0) = \Delta x(T)$ and $\Delta x(0) = -\Delta x(T)$, respectively. As $g = 0$, in these four cases, condition (2.1.47) is automatically satisfied with any $\theta \in \mathbb{R}$ and $C_1 = 0$.

Also, if $\alpha,\ \beta > 0$ are given, then with g defined by

$$g(z) := \frac{1}{p(0)} \frac{|z_1|^{p(0)}}{\alpha^{p(0)-1}} + \frac{1}{p(T)} \frac{|z_2|^{p(T)}}{\beta^{p(T)-1}}, \quad (\forall)\ z = (z_1, z_2) \in \mathbb{R} \times \mathbb{R}$$

and $K = \mathbb{R} \times \mathbb{R}$, we deduce the *Sturm-Liouville* type boundary conditions

$$x(0) - \alpha \Delta x(0) = 0, \quad x(T+1) + \beta \Delta x(T) = 0.$$

In this case, (2.1.47) is fulfilled with any $\theta \geq \min\{p(0), p(T)\}$ and $C_1 = 0$.

Consequently, an unified approach of all those boundary conditions is provided by means of (2.1.46) and sufficient conditions ensuring the existence of nontrivial solutions of (2.1.26) subjected to one of the above classical conditions can be easily stated by Theorem 2.1.14.

Remark 2.1.16. In the light of Remark 2.1.15, Theorem 4.1 proved in Y. Tian and W. Ge [125] for $p =$ constant, is a simple consequence of Theorem 2.1.14 by taking $g = 0$ and $j = I_K$, with $K = \mathbb{R} \times \mathbb{R}$.

2.1.4 Some remarks on constant λ_1

If $p =$ constant, it is easy to see that λ_1 from (2.1.19) becomes

$$\lambda_1 = \inf \left\{ \frac{|x|_1^p}{|x|_2^p} : x \in X_{PO},\ |x|_2 > 0,\ (x(0), x(T+1)) \in D(j) \right\},$$

where $|\cdot|_1$ and $|\cdot|_2$ denotes the semi-norms on X_{PO} defined by

$$|x|_1 = \left(\sum_{k=1}^{T+1} |\Delta x(k-1)|^p \right)^{\frac{1}{p}}, \quad |x|_2 = \left(\sum_{k=1}^{T} |x(k)|^p \right)^{\frac{1}{p}}.$$

We have the following:

Proposition 2.1.17. *Let $D(j)$ be a closed cone. If*

$$D(j) \cap \{(x, x),\ x \in \mathbb{R}\} = \{(0, 0)\},$$

then $\lambda_1 > 0$.

Proof. We use the idea from the proof of Proposition 3.6 in [76]. With this aim, X_{PO} will be endowed with the norm $\|x\| = (|x|_1^p + |x|_2^p)^{\frac{1}{p}}$. Suppose, by contradiction, that $\lambda_1 = 0$. Then, there is a sequence $\{x_n\} \subset X_{PO}$, $|x_n|_2 > 0$ such that $(x_n(0), x_n(T+1)) \in D(j)$ and

$$\frac{|x_n|_1^p}{|x_n|_2^p} < \frac{1}{n}, \quad (\forall)\ n \in \mathbb{N}. \tag{2.1.48}$$

Since $D(j)$ is a cone, clearly

$$y_n := \frac{x_n}{|x_n|_2} \in D(j), \quad (\forall)\ n \in \mathbb{N}.$$

Also, from (2.1.48), one has that $|y_n|_1 \to 0$, as $n \to \infty$ and since $\|y_n\|^p = |y_n|_1^p + 1$, we infer that $\{y_n\}$ is bounded in X_{PO}. Hence, there is a subsequence of $\{y_n\}$, still denoted by $\{y_n\}$ and some $y \in X_{PO}$ with $y_n \to y$, as $n \to \infty$. Since $D(j)$ is closed, we have $(y(0), y(T+1)) \in D(j)$. We obtain

$$|y|_2^p = \lim_{n\to\infty} |y_n|_2^p = 1$$

and so,

$$|y|_1^p + |y|_2^p = \|y\|^p \leq \liminf_{n\to\infty} \|y_n\|^p = 1.$$

Consequently, $|y|_1 = 0$ and hence, $y(k) = c$, for all $k \in \mathbb{Z}[0, T+1]$, with some $c \in \mathbb{R} \setminus \{0\}$. Therefore, it follows $(c, c) \in D(j)$, which is a contradiction. ■

Note that, if p = constant, then on account of Proposition 2.1.17, in the Neumann, periodic and Sturm-Liouville cases, $\underline{r}$ must be > 0, meaning $r > 0$ on $\mathbb{Z}[1, T]$, while in the cases of Dirichlet and antiperiodic boundary conditions $\underline{r}$ is allowed to be $= 0$ and hence, r may be ≥ 0 on $\mathbb{Z}[1, T]$.

However, for $p(\cdot)$ variable, λ_1 defined in (2.1.19) can be $= 0$ even in some simple situations, as when $D(j) = \{(0, 0)\}$ – the case of Dirichlet boundary conditions. For instance, let $T = 2$, $p(0) = p(2) = 2$, $p(1) = 3$, $x(0) = x(3) = 0$ and $x(1) = x(2) = 1$. Thus, for all $k \in \mathbb{Z}[0, 3]$, taking $x_a(k) = ax(k)$ $(a \in \mathbb{R} \setminus \{0\})$, one has

$$\frac{\sum_{k=1}^{3} |\Delta x_a(k-1)|^{p(k-1)}}{\sum_{k=1}^{2} |x_a(k)|^{p(k)}} = \frac{2a^2}{|a|^3 + a^2} \to 0, \quad \text{as } |a| \to \infty,$$

showing that $\lambda_1 = 0$. We emphasize that the same fact occurs in the continuous case for $p(x)$ variable – see X.L. Fan and D. Zhao [63].

In this view, the constant λ_1 from (2.1.19) seems to be not optimal, because from Theorem 2.1.5 we infer that the Dirichlet problem

$$\begin{cases} -\Delta_{p(k-1)} x(k-1) = f(k, x(k)), \quad (\forall)\ k \in \mathbb{Z}[1, T], \\ \\ x(0) = x(T+1) = 0, \end{cases}$$

is solvable provided that

$$\limsup_{|t| \to \infty} \frac{F(k, t)}{|t|^{p(k)}} < 0,$$

while in some cases the above "lim sup" can exceed 0 and the problem still remains solvable (see Theorem 5 from M. Galewski and S. Głąb [67]). Therefore, finding the optimal λ_1 remains an open problem; a key ingredient seems to be a Poincaré type inequality as obtained in P.G. Ciarlet and G. Dincă [46], P. Jebelean and R. Precup [81] or F.Y. Maeda [97] for Sobolev spaces with constant and/or variable exponent.

2.2 Existence of solutions for periodic and Neumann problems

In this section, we study the existence of solutions for equations of type (2.1.1) subjected to periodic and Neumann boundary conditions, i.e.,

$$\begin{cases} -\Delta_{p(k-1)}x(k-1) = f(k, x(k)), \quad (\forall)\ k \in \mathbb{Z}[1, T], \\ \\ x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T), \end{cases} \tag{2.2.1}$$

respectively,

$$\begin{cases} -\Delta_{p(k-1)}x(k-1) = f(k, x(k)), \quad (\forall)\ k \in \mathbb{Z}[1, T], \\ \\ \Delta x(0) = 0 = \Delta x(T), \end{cases} \tag{2.2.2}$$

where $\Delta_{p(\cdot)}$ is given in (2.1) and $f : \mathbb{Z}[1, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function. The solutions which we obtain appear as minimizers or saddle points of the corresponding energy functional.

From now on, we assume that the variable exponent p satisfies

$$p(0) = p(T) \tag{2.2.3}$$

when we refer to the periodic problem (2.2.1). We shall prove in detail the existence of solutions in the periodic case and only discuss the manner in which these results are adapted for the Neumann problem (2.2.2), since are obtained by no longer than 'mutatis mutandis' arguments.

The section is organized as follows. The functional framework and the variational setting are described in Paragraph 2.2.1. The main existence results for problems (2.2.1) and (2.2.2) without convexity assumptions are presented in Paragraphs 2.2.2 and 2.2.3. The case when the potential of the nonlinear term f has convexity properties is discussed in Paragraph 2.2.4. As applications, we prove in the last paragraph of the section upper and lower solutions theorems for both of the problems.

The results from this section are obtained in [18]. We emphasize that, in the periodic case, the existence results proved in [84] for p = constant are immediate consequences of the corresponding ones which will be presented in this section.

2.2.1 The variational setting

To treat the periodic problem (2.2.1) we introduce the spaces

$$X_P := \{x : \mathbb{Z}[0, T+1] \to \mathbb{R} \mid x(0) = x(T+1)\}$$

and

$$X_{P,0} := \left\{ \tilde{x} \in X_P \;\Big|\; \sum_{k=1}^{T} \tilde{x}(k) = 0 \right\},$$

while in the case of Neumann problem (2.2.2), we shall use

$$X_N := \{x : \mathbb{Z}[0, T+1] \to \mathbb{R}\} \;\; (= X_{PO})$$

and

$$X_{N,0} := \left\{ \tilde{x} \in X_N \;\Big|\; \sum_{k=1}^{T} \tilde{x}(k) = 0 \right\}.$$

Further, for the convenience in notations we generically denote by X one of the spaces X_P or X_N and X_0 will stand for $X_{P,0}$ and $X_{N,0}$, respectively.

For each $x \in X$ we set $\overline{\overline{x}} := (1/T)\sum_{k=1}^{T} x(k) \in \mathbb{R}$ and $\tilde{x} := x - \overline{\overline{x}} \in X_0$. This enables us to split $X = \mathbb{R} \oplus X_0$. The space X_0 will be considered with the following Luxemburg type norm (see Proposition 2.1.1)

$$\|\tilde{x}\|_{p(\cdot)} = \inf\left\{ \nu > 0 \;:\; \sum_{k=1}^{T+1} \frac{1}{p(k-1)} \left| \frac{\Delta \tilde{x}(k-1)}{\nu} \right|^{p(k-1)} \leq 1 \right\}, \quad (\forall)\ \tilde{x} \in X_0,$$

while X will be endowed with the norm

$$\|x\|_X = |\overline{\overline{x}}\,| + \|\tilde{x}\|_{p(\cdot)}, \quad (\forall)\ x = \overline{\overline{x}} + \tilde{x} \in \mathbb{R} \oplus X_0.$$

Since X_0 and X are finite dimensional spaces, any norm on X_0 (resp. X) is equivalent with $\|\cdot\|_{p(\cdot)}$ (resp. $\|\cdot\|_X$). Also, we shall invoke the usual sup-norm defined in (2.1.4).

Let $\varphi_X : X \to \mathbb{R}$ be defined by

$$\varphi_X(x) = \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta x(k-1)|^{p(k-1)}, \quad (\forall)\ x \in X$$

(also see (2.1.5)). We have that $\varphi_X \in C^1(X, \mathbb{R})$ and its derivative is given in (2.1.6), with X instead of X_{PO}. As in Proposition 2.1.2, for $\tilde{x} \in X_0$, the following hold true

$$\begin{aligned} &\|\tilde{x}\|_{p(\cdot)}^{p^-} \leq \varphi_X(\tilde{x}) \leq \|\tilde{x}\|_{p(\cdot)}^{p^+}, \;\; \text{if } \|\tilde{x}\|_{p(\cdot)} > 1; \\ &\|\tilde{x}\|_{p(\cdot)}^{p^+} \leq \varphi_X(\tilde{x}) \leq \|\tilde{x}\|_{p(\cdot)}^{p^-}, \;\; \text{if } \|\tilde{x}\|_{p(\cdot)} < 1. \end{aligned} \tag{2.2.4}$$

Now, for given $\ell \in \mathbb{R}^{\mathbb{Z}[1,T]}$, we define $E_X : X \to \mathbb{R}$ by

$$E_X(x) = \varphi_X(x) - \sum_{k=1}^{T} \ell(k)x(k), \quad (\forall)\ x \in X. \tag{2.2.5}$$

Note that $E_X \in C^1(X, \mathbb{R})$ and

$$\langle E'_X(x), y\rangle = \sum_{k=1}^{T+1} h_{p(k-1)}(\Delta x(k-1))\Delta y(k-1) - \sum_{k=1}^{T} \ell(k)y(k), \quad (\forall)\ x, y \in X. \tag{2.2.6}$$

Next, let us consider the following simple periodic problem

$$\begin{cases} -\Delta_{p(k-1)}x(k-1) = \ell(k), \quad (\forall)\ k \in \mathbb{Z}[1, T], \\ x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T). \end{cases} \tag{2.2.7}$$

Proposition 2.2.1. *Assume that* (2.2.3) *holds true. Then, a function $x \in X_P$ is a solution of* (2.2.7) *if and only if it is a critical point of E_{X_P}.*

Proof. Taking into account the computation in (2.1.10), we have

$$\begin{aligned} \langle E'_{X_P}(x), y\rangle = {} & h_{p(T)}(\Delta x(T))y(T+1) - h_{p(0)}(\Delta x(0))y(0) \\ & - \sum_{k=1}^{T} \left(\Delta_{p(k-1)}x(k-1) + \ell(k)\right) y(k), \end{aligned} \tag{2.2.8}$$

for all $x, y \in X_P$. From (2.2.3) and (2.2.8), if x is a solution of problem (2.2.7), then it is obvious that it is a critical point of E_{X_P}.

To prove the converse implication, let $x \in X_P$ be so that $\langle E'_{X_P}(x), y\rangle = 0$ for all $y \in X_P$. Again, by (2.2.8) it follows

$$\sum_{k=1}^{T} \left(\Delta_{p(k-1)}x(k-1) + \ell(k)\right) y(k) = 0,$$

for all $y \in X_P$ with $y(0) = 0 = y(T+1)$. This yields

$$-\Delta_{p(k-1)}x(k-1) = \ell(k), \quad (\forall)\ k \in \mathbb{Z}[1, T]. \tag{2.2.9}$$

We infer

$$h_{p(T)}(\Delta x(T))y(T+1) = h_{p(0)}(\Delta x(0))y(0), \quad (\forall)\ y \in X_P.$$

By virtue of (2.2.3), taking $y \in X_P$ with $y(0) = 1 = y(T+1)$, we get

$$\Delta x(T) = \Delta x(0). \tag{2.2.10}$$

From (2.2.9) and (2.2.10), we conclude that x is a solution of problem (2.2.7). ■

Now, let $\mathcal{F}_X$ be the C^1 functional on X, defined by

$$\mathcal{F}_X(x) = \sum_{k=1}^{T} F(k, x(k)), \quad (\forall)\ x \in X. \tag{2.2.11}$$

We recall that F is the primitive of f with respect to the second variable (see (2.1.13)) and the derivative of $\mathcal{F}_X$ is

$$\langle \mathcal{F}_X'(x), y\rangle = \sum_{k=1}^{T} f(k, x(k))y(k), \quad (\forall)\ x, y \in X. \tag{2.2.12}$$

The energy functional corresponding to problem (2.2.1) (resp. (2.2.2)) is

$$\mathcal{E}_X(x) = \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta x(k-1)|^{p(k-1)} - \sum_{k=1}^{T} F(k, x(k)), \quad (\forall)\ x \in X,$$

with $X = X_P$ (resp. $X = X_N$). Clearly,

$$\mathcal{E}_X(x) = \varphi_X(x) - \mathcal{F}_X(x),$$

therefore $\mathcal{E}_X \in C^1(X, \mathbb{R})$ and

$$\langle \mathcal{E}_X'(x), y\rangle = \sum_{k=1}^{T+1} h_{p(k-1)}(\Delta x(k-1))\Delta y(k-1) - \sum_{k=1}^{T} f(k, x(k))y(k), \tag{2.2.13}$$

for all $x, y \in X$.

Proposition 2.2.2. *Assume that* (2.2.3) *holds true. A function $x \in X_P$ is solution of problem* (2.2.1) *if and only if it is a critical point of $\mathcal{E}_{X_P}$.*

Proof. Viewing (2.2.13), Proposition 2.2.1 applies with $\ell(k) = f(k, x(k))$ for all $k \in \mathbb{Z}[1, T]$. ■

Using the Neumann problem

$$\begin{cases} -\Delta_{p(k-1)} x(k-1) = \ell(k), \quad (\forall)\ k \in \mathbb{Z}[1, T], \\ \\ \Delta x(0) = 0 = \Delta x(T), \end{cases} \tag{2.2.14}$$

instead of (2.2.7) and exactly the same strategy as above, but with E_{X_N} (resp. $\mathcal{E}_{X_N}$) instead of E_{X_P} (resp. $\mathcal{E}_{X_P}$), we obtain the following

Proposition 2.2.3. *A function $x \in X_N$ is solution of problem* (2.2.14) *if and only if it is a critical point of E_{X_N}.*

Proposition 2.2.4. *A function $x \in X_N$ is solution of problem* (2.2.2) *if and only if it is a critical point of $\mathcal{E}_{X_N}$.*

Remark 2.2.5. Note that assumption (2.2.3) is not required in the case of Neumann boundary conditions.

2.2.2 Bounded nonlinearities

We begin by a simple result concerning the solvability of problem (2.2.7). First, we introduce

$$\mathcal{L} := \left\{ \ell \in \mathbb{R}^{\mathbb{Z}[1,T]} \;\Big|\; \sum_{k=1}^{T} \ell(k) = 0 \right\}$$

and by virtue of (2.2.6), it is easy to check that if (2.2.7) is solvable then necessarily one has $\ell \in \mathcal{L}$. Actually, this is also a sufficient condition, as shown in the next proposition.

Proposition 2.2.6. *If $\ell \in \mathcal{L}$ and* (2.2.3) *holds true, then problem* (2.2.7) *has a unique solution $\tilde{x}_0 \in X_{P,0}$. Moreover, $\tilde{x}_0$ is the unique minimizer of E_{X_P} on $X_{P,0}$.*

Proof. First, we prove that E_{X_P} is coercive on $(X_{P,0}, \|\cdot\|_{p(\cdot)})$. For this, let $\tilde{x} \in X_{P,0}$ be with $\|\tilde{x}\|_{p(\cdot)} > 1$. Using (2.2.4), we estimate E_{X_P} as follows

$$E_{X_P}(\tilde{x}) \geq \|\tilde{x}\|_{p(\cdot)}^{p^-} - \sum_{k=1}^{T} |\ell(k)\tilde{x}(k)| \geq \|\tilde{x}\|_{p(\cdot)}^{p^-} - \|\ell\|_\infty \sum_{k=0}^{T+1} |\tilde{x}(k)|$$

and since any norm on $X_{P,0}$ is equivalent with $\|\cdot\|_{p(\cdot)}$, there is a constant $C_1 > 0$ so that

$$E_{X_P}(\tilde{x}) \geq \|\tilde{x}\|_{p(\cdot)}^{p^-} - C_1 \|\ell\|_\infty \|\tilde{x}\|_{p(\cdot)} \to +\infty, \quad \text{as } \|\tilde{x}\|_{p(\cdot)} \to \infty.$$

This ensure that there is some $\tilde{x}_0 \in X_{P,0}$ with

$$E_{X_P}(\tilde{x}_0) = \inf\left\{E_{X_P}(\tilde{x}) \mid \tilde{x} \in X_{P,0}\right\}.$$

From (2.2.6), we get

$$\sum_{k=1}^{T+1} h_{p(k-1)}(\Delta\tilde{x}_0(k-1))\Delta\tilde{y}(k-1) = \sum_{k=1}^{T} \ell(k)\tilde{y}(k), \quad (\forall)\ \tilde{y} \in X_{P,0} \quad (2.2.15)$$

Let $y \in X_P$ be arbitrarily chosen. Taking $\tilde{y} = y - (1/T)\sum_{k=1}^{T} y(k)$ in

(2.2.15) and because $\ell \in \mathcal{L}$, one obtains

$$\sum_{k=1}^{T+1} h_{p(k-1)}(\Delta\tilde{x}_0(k-1))\Delta y(k-1) = \sum_{k=1}^{T} \ell(k)y(k), \quad (\forall)\ y \in X_P,$$

which means that $\tilde{x}_0$ is a solution of problem (2.2.7) (see Proposition 2.2.1).

To prove the uniqueness of $\tilde{x}_0$, let $\tilde{z}_0$ be another solution of problem (2.2.1). From (2.2.15), we have

$$\sum_{k=1}^{T+1} [h_{p(k-1)}(\Delta\tilde{x}_0(k-1)) - h_{p(k-1)}(\Delta\tilde{z}_0(k-1))]\Delta\tilde{y}(k-1) = 0,$$

for all $\tilde{y} \in X_{P,0}$. Hence, taking $\tilde{y} = \tilde{x}_0 - \tilde{z}_0$ in the previous equality and on account of the strict monotonicity of $h_{p(\cdot)}$, we deduce that $\Delta\tilde{x}_0(k-1) = \Delta\tilde{z}_0(k-1)$ ($k \in \mathbb{Z}[1, T+1]$), which implies that $\tilde{y}$ is a constant. But $\tilde{y} \in X_{P,0}$ and so $\tilde{y} = 0$, i.e. $\tilde{x}_0 = \tilde{z}_0$. Also, assuming that E_{X_P} has a second minimizer $\tilde{z}_0$ on $X_{P,0}$, then $\tilde{z}_0$ is a solution of problem (2.2.1). But we just showed that (2.2.1) has a unique solution and so $\tilde{x}_0 = \tilde{z}_0$, which complete the proof. ■

Next, for given $\ell \in \mathcal{L}$, we consider the slightly more general version of problem (2.2.1):

$$\begin{cases} -\Delta_{p(k-1)}x(k-1) = f(k, x(k)) + \ell(k), \quad (\forall)\ k \in \mathbb{Z}[1, T], \\ \\ x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T). \end{cases} \tag{2.2.16}$$

Instead of $\mathcal{E}_{X_P}$, the energy functional associated with problem (2.2.16) will be $\widetilde{\mathcal{E}}_{X_P} : X_P \to \mathbb{R}$, defined by

$$\widetilde{\mathcal{E}}_{X_P}(x) = E_{X_P}(\tilde{x}) - \mathcal{F}_{X_P}(x), \quad (\forall)\ x = \overline{\overline{x}} + \tilde{x} \in \mathbb{R} \oplus X_{P,0}, \tag{2.2.17}$$

with E_{X_P} in (2.2.5).

In the next theorem we consider periodic nonlinearities of pendulum type. The corresponding classical result is due to G. Hamel [73].

Theorem 2.2.7. *Assume that* (2.2.3) *holds true and that there exists* $\omega > 0$ *such that* $F(k, t+\omega) = F(k, t)$, *for all* $k \in \mathbb{Z}[1, T]$ *and* $t \in \mathbb{R}$. *Then, for each* $\ell \in \mathcal{L}$, *the periodic problem* (2.2.16) *has at least one solution which minimizes* $\widetilde{\mathcal{E}}_{X_P}$ *on* X_P.

Proof. Since F is continuous and periodic with respect to the second variable, there exists $M > 0$ such that

$$|F(k, t)| \le M, \quad (\forall)\ k \in \mathbb{Z}[1, T],\ t \in \mathbb{R}.$$

Let $\tilde{x}_0 \in X_{P,0}$ be the unique solution of problem (2.2.7) (see Proposition 2.2.6). We infer

$$\widetilde{\mathcal{E}}_{X_P}(x) \geq E_{X_P}(\tilde{x}_0) - TM, \quad (\forall)\ x \in X_P$$

and hence, $\widetilde{\mathcal{E}}_{X_P}$ is bounded from below. We show that $\widetilde{\mathcal{E}}_{X_P}$ admits a bounded minimizing sequence, which will conclude the proof.

Let $\{x_n\} \subset X_P$ be a minimizing sequence for $\widetilde{\mathcal{E}}_{X_P}$. For each $n \in \mathbb{N}$, there exists $m_n \in \mathbb{Z}$ such that $\overline{\overline{x}}_n + m_n\omega \in [0,\omega]$. Setting $y_n = x_n + m_n\omega$ we have $\tilde{y}_n = y_n - (1/T)\sum_{k=1}^{T} y_n(k) = \tilde{x}_n$ and $\overline{\overline{y}}_n = y_n - \tilde{y}_n = \overline{\overline{x}}_n + m_n\omega \in [0,\omega]$. By the periodicity of $F(k,\cdot)$ we obtain that $\widetilde{\mathcal{E}}_{X_P}(x_n) = \widetilde{\mathcal{E}}_{X_P}(y_n)$ and thus $\{y_n\}$ is also a minimizing sequence for $\widetilde{\mathcal{E}}_{X_P}$.

Now, using that $\{\widetilde{\mathcal{E}}_{X_P}(y_n)\}$ is bounded and the equivalence of the norms on $X_{P,0}$, we can find constants $M_1, C_1 > 0$ such that

$$M_1 \geq \widetilde{\mathcal{E}}_{X_P}(y_n) \geq \|\tilde{y}_n\|_{p(\cdot)}^{p^-} - C_1\|\ell\|_\infty \|\tilde{y}_n\|_{p(\cdot)} - TM,$$

provided that $\|\tilde{y}_n\|_{p(\cdot)} > 1$ (see (2.2.4)). Hence, $\{\|\tilde{y}_n\|_{p(\cdot)}\}$ is bounded and then, since $\|y_n\|_{X_P} = |\overline{\overline{y}}_n| + \|\tilde{y}_n\|_{p(\cdot)}$ and $\overline{\overline{y}}_n \in [0,\omega]$, we infer that $\{y_n\}$ is bounded in X_P. ■

Remark 2.2.8. The ω-periodicity assumption on the primitive F in the previous theorem is equivalent with $\int_0^\omega f(k,\tau)d\tau = 0$ and $f(k,t+\omega) = f(k,t)$ for all $k \in \mathbb{Z}[1,T]$ and $t \in \mathbb{R}$. Indeed, if f is ω-periodic with respect to the second variable, then

$$(F(k,t+\omega) - F(k,t))'_t = f(k,t+\omega) - f(k,t) = 0, \quad (\forall)\ k \in \mathbb{Z}[1,T],\ t \in \mathbb{R}.$$

Hence, there exists $c_k \in \mathbb{R}$ such that $F(k,t+\omega) - F(k,t) = c_k$, for every $k \in \mathbb{Z}[1,T]$ and $t \in \mathbb{R}$. Taking $t = 0$, we get

$$c_k = F(k,0+\omega) - F(k,0) = F(k,\omega) = \int_0^\omega f(k,\tau)d\tau = 0, \quad (k \in \mathbb{Z}[1,T]).$$

Thus, $F(k,t+\omega) = F(k,t)$ for all $k \in \mathbb{Z}[1,T]$ and $t \in \mathbb{R}$. Conversely, it is clear that the ω-periodicity of the potential F implies the ω-periodicity of f and $\int_t^{t+\omega} f(k,\tau)d\tau = 0$, for all $k \in \mathbb{Z}[1,T]$ and $t \in \mathbb{R}$. Then, taking $t = 0$, one has that $\int_0^\omega f(k,\tau)d\tau = 0$.

Theorem 2.2.9. *Assume that* (2.2.3) *holds true and f is bounded. If either*

$$\sum_{k=1}^{T} F(k,t) \to -\infty, \qquad \textit{as } |t| \to \infty \tag{2.2.18}$$

or

$$\sum_{k=1}^{T} F(k,t) \to +\infty, \qquad as\ |t| \to \infty, \tag{2.2.19}$$

then problem (2.2.1) *has at least one solution* $x_0 \in X_P$. *Moreover, if* (2.2.18) *holds true then* x_0 *minimizes* $\mathcal{E}_{X_P}$ *on* X_P.

Proof. If (2.2.18) holds true, we shall prove that $\mathcal{E}_{X_P}$ is coercive on X_P and then, arguing as in the proof of Theorem 2.1.5, by the direct method in the calculus of variations and Proposition 2.2.2, problem (2.2.1) has at least one solution which minimizes $\mathcal{E}_{X_P}$ on X_P. In the case when (2.2.19) is fulfilled, we shall apply the Saddle Point Theorem (see Theorem 1.8).

Since f is bounded, there exists $M > 0$ such that

$$|f(k,t)| \leq M, \quad (\forall)\ k \in \mathbb{Z}[1,T],\ t \in \mathbb{R}. \tag{2.2.20}$$

For $x \in X_P$ we write $x = \overline{\overline{x}} + \tilde{x}$, where $\overline{\overline{x}} = (1/T)\sum_{k=1}^{T} x(k)$. Then, using (2.2.4) and (2.2.20), we estimate $\mathcal{E}_{X_P}$ as follows

$$\begin{aligned}
\mathcal{E}_{X_P}(x) &= \varphi_{X_P}(\tilde{x}) - \sum_{k=1}^{T} F(k,\overline{\overline{x}}) - \sum_{k=1}^{T} \big(F(k,x(k)) - F(k,\overline{\overline{x}})\big) \\
&= \varphi_{X_P}(\tilde{x}) - \sum_{k=1}^{T} F(k,\overline{\overline{x}}) - \sum_{k=1}^{T} \int_0^1 f(k,\overline{\overline{x}} + s\tilde{x}(k))\tilde{x}(k)ds \\
&\geq \|\tilde{x}\|_{p(\cdot)}^{p^-} - \sum_{k=1}^{T} F(k,\overline{\overline{x}}) - M\sum_{k=0}^{T+1} |\tilde{x}(k)|, \text{ for } \|\tilde{x}\|_{p(\cdot)} > 1.
\end{aligned} \tag{2.2.21}$$

Taking into account the equivalence of the norms on $X_{P,0}$, one obtains

$$\mathcal{E}_{X_P}(x) \geq \|\tilde{x}\|_{p(\cdot)}^{p^-} - C_1\|\tilde{x}\|_{p(\cdot)} - \sum_{k=1}^{T} F(k,\overline{\overline{x}}), \quad (\forall)\ x \in X_P,\ \|\tilde{x}\|_{p(\cdot)} > 1, \tag{2.2.22}$$

with C_1 a positive constant.

Now, let $\{x_n\} \subset X_P$ be a sequence such that $\|x_n\|_{X_P} \to \infty$. Arguing by contradiction, we shall show that $\mathcal{E}_{X_P}(x_n) \to +\infty$. Suppose that $\mathcal{E}_{X_P}(x_n) \not\to +\infty$, i.e. there exists $M_1 > 0$ such that, for all $n_0 \in \mathbb{N}$, we can find $n \geq n_0$ so that $\mathcal{E}_{X_P}(x_n) \leq M_1$. Therefore, taking

$n_0 = 1,$ there exists $n_1 \geq 1$ such that $\mathcal{E}_{X_P}(x_{n_1}) \leq M_1$;

$n_0 = n_1 + 1,$ there exists $n_2 \geq n_1$ such that $\mathcal{E}_{X_P}(x_{n_2}) \leq M_1$;

$n_0 = n_2 + 1,$ there exists $n_3 \geq n_2$ such that $\mathcal{E}_{X_P}(x_{n_3}) \leq M_1$;

$\ldots$

So, there is a subsequence of $\{x_n\}$, still denoted by $\{x_n\}$ such that $\mathcal{E}_{X_P}(x_n) \leq M_1$, for all $n \in \mathbb{N}$. Since $\|x_n\|_{X_P} = |\overline{\overline{x}}_n| + \|\tilde{x}_n\|_{p(\cdot)} \to \infty$, at least one of the sequences $\{\|\tilde{x}_n\|_{p(\cdot)}\}$ or $\{|\overline{\overline{x}}_n|\}$ is unbounded.

Suppose that $\|\tilde{x}_n\|_{p(\cdot)}$ is unbounded, i.e. there exists a subsequence $\{\|\tilde{x}_{n_m}\|_{p(\cdot)}\} \subset \{\|\tilde{x}_n\|_{p(\cdot)}\}$ so that $\|\tilde{x}_{n_m}\|_{p(\cdot)} \to \infty$ as $m \to \infty$. If $\{|\overline{\overline{x}}_{n_m}|\}$ is bounded, then $\sum_{k=1}^{T} F(k, \overline{\overline{x}}_{n_m})$ is also bounded, which together with (2.2.22) leads to a contradiction. If $\{|\overline{\overline{x}}_{n_m}|\}$ is unbounded, then there exists a subsequence $\{|\overline{\overline{x}}_{n_{m_l}}|\} \subset \{|\overline{\overline{x}}_{n_m}|\}$ such that $|\overline{\overline{x}}_{n_{m_l}}| \to \infty$ when $l \to \infty$. Also, $\|\tilde{x}_{n_{m_l}}\|_{p(\cdot)} \to \infty$ as $l \to \infty$ and by virtue of (2.2.18) and (2.2.22), we obtain again a contradiction.

Similarly, in the case when $\{|\overline{\overline{x}}_n|\}$ is unbounded, arguing as above, we get again a contradiction. Consequently,

$$\mathcal{E}_{X_P}(x) \to +\infty, \qquad \text{as } \|x\|_{X_P} \to \infty,$$

i.e. $\mathcal{E}_{X_P}$ is coercive on X_P and the conclusion follows.

If (2.2.19) holds, then $\mathcal{E}_{X_P}$ is neither bounded from below, nor from above. Indeed, if $x = c \in \mathbb{R}$ is a constant function then

$$\mathcal{E}_{X_P}(c) = -\sum_{k=1}^{T} F(k, c) \to -\infty, \qquad \text{as } |c| \to \infty, \tag{2.2.23}$$

while, if $\tilde{x} \in X_{P,0}$, then by (2.2.22), we get

$$\mathcal{E}_{X_P}(\tilde{x}) \to +\infty, \qquad \text{as } \|\tilde{x}\|_{p(\cdot)} \to \infty. \tag{2.2.24}$$

From (2.2.23) and (2.2.24), there exists some $R > 0$ such that

$$\sup_{S_R} \mathcal{E}_{X_P} < \inf_{X_{P,0}} \mathcal{E}_{X_P},$$

where $S_R = \{x \in \mathbb{R} \mid |x| = R\}$. It remains to show that $\mathcal{E}_{X_P}$ satisfies the (PS) condition, which will conclude the proof.

Let $\{x_n\} \subset X_P$ be a sequence for which $\{\mathcal{E}_{X_P}(x_n)\}$ is bounded and

$$\mathcal{E}'_{X_P}(x_n) \to 0, \qquad \text{as } n \to \infty. \tag{2.2.25}$$

Since X_P is finite dimensional, we have to prove that $\{x_n\}$ is bounded. From (2.2.25), there is $n_0 \in \mathbb{N}$ such that

$$|\langle \mathcal{E}'_{X_P}(x_n), z \rangle| \leq \|z\|_{X_P},$$

for all $n \geq n_0$ and $z \in X_P$.

Thus, we infer that

$$|\langle \mathcal{E}'_{X_P}(x_n), \tilde{x}_n \rangle| = \left| \sum_{k=1}^{T+1} |\Delta \tilde{x}_n(k-1)|^{p(k-1)} - \sum_{k=1}^{T} f(k, x_n(k))\tilde{x}_n(k) \right| \le \|\tilde{x}_n\|_{p(\cdot)}.$$

This, together with (2.2.4), (2.2.20) and the equivalence of the norms on $X_{P,0}$, implies

$$p^- \|\tilde{x}_n\|_{p(\cdot)}^{p^-} - \|\tilde{x}_n\|_{p(\cdot)} \le MC_2 \|\tilde{x}_n\|_{p(\cdot)}, \quad (\forall)\ n \ge n_0,\ \text{if } \|\tilde{x}_n\|_{p(\cdot)} > 1,$$

with $C_2 > 0$ a constant and hence we get that $\{\|\tilde{x}_n\|_{p(\cdot)}\}$ is bounded.

On the other hand, using (2.2.4), (2.2.21) and the fact that $\{\mathcal{E}_{X_P}(x_n)\}$ is bounded, we can find $C_3 \in \mathbb{R}$ so that

$$\|\tilde{x}_n\|_{p(\cdot)}^{p^+} - \sum_{k=1}^{T} F(k, \overline{x}_n) - \sum_{k=1}^{T} \int_0^1 f(k, \overline{x}_n + s\tilde{x}_n(k))\tilde{x}_n(k) ds \ge C_3, \quad (\forall)\ n \in \mathbb{N}$$

and then, by the boundedness of the sequence $\{\|\tilde{x}_n\|_{p(\cdot)}\}$ and condition (2.2.20), we have that

$$\sum_{k=1}^{T} F(k, \overline{x}_n) \le C_4, \quad (\forall)\ n \in \mathbb{N}.$$

Hence, by (2.2.19) we obtain that $\{|\overline{x}_n|\}$ is bounded. Consequently, $\{x_n\}$ is bounded in X_P and the proof is complete. ■

Remark 2.2.10. Condition (2.2.19) is of the type introduced by S. Ahmad, A.C. Lazer and J.L. Paul in [4] for the Laplacian with Dirichlet boundary conditions. Also, (2.2.18) appears as being 'dual' to condition (2.2.19).

Example 2.2.11. [see Example 3.6 in [84] for $p =$ constant] The problem

$$\begin{cases} -\Delta_{p(k-1)} x(k-1) = \sin(a \text{ sign } x(k) - x(k)) - \sin(a \text{ sign } x(k)) + \ell(k), \\ \\ x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T) \end{cases}$$

is solvable for any $a \in \mathbb{R}$ and $\ell \in \mathcal{L}$. Indeed, for $a \in \mathbb{R} \setminus \{m\pi \mid m \in \mathbb{Z}\}$, a straightforward computation shows that

$$F(k, x) = -|x| \sin a + \cos(a - |x|) - \cos a + \ell(k)x$$

and the conclusion follows from Theorem 2.2.9. Also, if $a \in \{m\pi \mid m \in \mathbb{Z}\}$, then Theorem 2.2.7 applies with

$$F(k, x) = \cos(a - |x|) - \cos a.$$

Remark 2.2.12. (i) Theorems 2.2.7 and 2.2.9 are discrete $p(\cdot)$-Laplacian variants of Theorem 4.7, Corollary 1.2 and Theorem 1.5 in [105]. Also, the previous example is a discrete $p(\cdot)$-Laplacian variant of the example at pages 13-14 in [105].

(ii) Theorem 3.3 (resp. Theorem 3.5) proved in [84] for $p =$ constant is an immediate consequence of Theorem 2.2.9 (resp. Theorem 2.2.7).

Using exactly the same strategy as above, in the case of Neumann boundary conditions, we have the following existence results.

Theorem 2.2.13. *Assume that there exists $\omega > 0$ such that $F(k, t + \omega) = F(k, t)$, for all $k \in \mathbb{Z}[1, T]$ and $x \in \mathbb{R}$. Then, for each $\ell \in \mathcal{L}$, the problem*

$$\begin{cases} -\Delta_{p(k-1)}x(k-1) = f(k, x(k)) + \ell(k), \quad (\forall)\ k \in \mathbb{Z}[1, T], \\ \\ \Delta x(0) = 0 = \Delta x(T) \end{cases} \tag{2.2.26}$$

has at least one solution which minimizes on X_N the corresponding Euler-Lagrange functional $\widetilde{\mathcal{E}}_{X_N} : X_N \to \mathbb{R}$, defined by

$$\widetilde{\mathcal{E}}_{X_N}(x) = E_{X_N}(\tilde{x}) - \mathcal{F}_{X_N}(x),$$

i.e.,

$$\widetilde{\mathcal{E}}_{X_N}(x) = \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta \tilde{x}(k-1)|^{p(k-1)} - \sum_{k=1}^{T} \ell(k)\tilde{x}(k) - \sum_{k=1}^{T} F(k, x(k)),$$

for all $x \in X_N$, $x = \overline{\overline{x}} + \tilde{x}$, where $\overline{\overline{x}} = (1/T)\sum_{k=1}^{T} x(k)$.

Theorem 2.2.14. *If f is bounded and either* (2.2.18) *or* (2.2.19) *is satisfied, then problem* (2.2.2) *has at least one solution $x_0 \in X_N$. Moreover, if* (2.2.18) *holds true then x_0 minimizes $\mathcal{E}_{X_N}$ on X_N.*

2.2.3 A comparison result

Theorem 2.2.15. *Assume* (2.2.3), *$\ell \in \mathcal{L}$ and let $\tilde{x}_0$ be the unique solution of problem* (2.2.7) *in $X_{P,0}$. If there exists a constant $M \geq 0$ such that*

$$F(k, t) \leq M, \quad (\forall)\ t \in \mathbb{R} \text{ and } k \in \mathbb{Z}[1, T] \tag{2.2.27}$$

and

$$\sum_{k=1}^{T} \limsup_{|t| \to \infty} F(k, t) < \sum_{k=1}^{T} F(k, \tilde{x}_0(k)), \tag{2.2.28}$$

then problem (2.2.16) *has at least one solution which minimizes* $\widetilde{\mathcal{E}}_{X_P}$ *on* X_P.

Proof. By Proposition 2.2.6, (2.2.17) and (2.2.27), we get

$$\widetilde{\mathcal{E}}_{X_P}(x) \geq E_{X_P}(\tilde{x}_0) - TM, \quad (\forall)\ x \in X_P$$

and hence, $\widetilde{\mathcal{E}}_{X_P}$ is bounded from below. It remains to show that $\widetilde{\mathcal{E}}_{X_P}$ admits a bounded minimizing sequence. Setting

$$c := \inf\left\{\widetilde{\mathcal{E}}_{X_P}(x) \mid x \in X_P\right\}\ (\in \mathbb{R}),$$

from (2.2.28), one has

$$\begin{aligned} c \leq \widetilde{\mathcal{E}}_{X_P}(\tilde{x}_0) &= E_{X_P}(\tilde{x}_0) - \sum_{k=1}^{T} F(k, \tilde{x}_0(k)) \\ &< E_{X_P}(\tilde{x}_0) - \sum_{k=1}^{T} \limsup_{|t|\to\infty} F(k,t). \end{aligned} \tag{2.2.29}$$

Now, let $\{x_n\} \subset X_P$ be a minimizing sequence for $\widetilde{\mathcal{E}}_{X_P}$, i.e.,

$$\widetilde{\mathcal{E}}_{X_P}(x_n) \to c, \qquad \text{as } n \to \infty. \tag{2.2.30}$$

There are constants $M_1, C_1 > 0$ such that

$$\begin{aligned} M_1 \geq \widetilde{\mathcal{E}}_{X_P}(x_n) &= E_{X_P}(\tilde{x}_n) - \sum_{k=1}^{T} F(k, x_n(k)) \\ &\geq \|\tilde{x}_n\|_{p(\cdot)}^{p^-} - C_1\|\ell\|_\infty \|\tilde{x}_n\|_{p(\cdot)} - TM, \quad \text{if } \|\tilde{x}_n\|_{p(\cdot)} > 1 \quad \text{(see (2.2.4))} \end{aligned}$$

which implies that $\{\tilde{x}_n\}$ is bounded in $(X_{P,0}, \|\cdot\|_{p(\cdot)})$ and hence, we can find $M_2 > 0$ so that

$$|\tilde{x}_n(k)| \leq M_2, \quad (\forall)\ k \in \mathbb{Z}[0, T+1],\ (\forall)\ n \in \mathbb{N}. \tag{2.2.31}$$

Suppose, by contradiction, that $\{x_n\}$ is not bounded in X_P. We may assume that it holds $\|x_n\|_{X_P} \to \infty$. As $\|\tilde{x}_n\|_{p(\cdot)}$ is bounded, it follows

$$|\overline{\overline{x}}_n| = \|x_n\|_{X_P} - \|\tilde{x}_n\|_{p(\cdot)} \to +\infty, \qquad \text{as } n \to \infty.$$

This, together with (2.2.31), implies

$$|x_n(k)| \geq |\overline{\overline{x}}_n| - |\tilde{x}_n(k)| \geq |\overline{\overline{x}}_n| - M_2 \to +\infty, \qquad \text{as } n \to \infty, \tag{2.2.32}$$

for all $k \in \mathbb{Z}[0, T+1]$.

Now, we shall show that

$$\limsup_{|t|\to\infty} F(k,t) \geq \limsup_{n\to\infty} F(k,x_n(k)), \quad (k\in\mathbb{Z}[1,T]). \tag{2.2.33}$$

First, since $|x_n(k)|\to\infty$ ($k\in\mathbb{Z}[0,T+1]$) as $n\to\infty$, for any $m>0$, there exists $n_m\in\mathbb{N}$ such that $|x_n(k)|\geq m$, for all $n\geq n_m$ and $k\in\mathbb{Z}[0,T+1]$. Therefore, taking

$$\begin{aligned}
m=1,\ & \text{there exists } n_1\in\mathbb{N} \text{ such that } |x_n(k)|\geq 1,\\
& \text{for all } n\geq n_1 \text{ and } k\in\mathbb{Z}[0,T+1];\\
m=2,\ & \text{there exists } n_2^0\in\mathbb{N} \text{ such that } |x_n(k)|\geq 2 \text{ for all } n\geq n_2^0\\
& \text{and setting } n_2=\max\{n_2^0,n_1\}+1>n_1,\ \text{one has}\\
& |x_n(k)|\geq 2,\ \text{for all } n\geq n_2 \text{ and } k\in\mathbb{Z}[0,T+1];\\
\dots&\\
m=l,\ & \text{there exists } n_l>n_{l-1} \text{ such that } |x_n(k)|\geq l,\\
& \text{for all } n\geq n_l \text{ and } k\in\mathbb{Z}[0,T+1].
\end{aligned}$$

So, for all $k\in\mathbb{Z}[1,T]$, we get

$$\begin{gathered}
\{F(k,x_n(k)) : n\geq n_1\}\subset\{F(k,t) : |t|\geq 1\};\\
\dots\\
\{F(k,x_n(k)) : n\geq n_l\}\subset\{F(k,t) : |t|\geq l\}
\end{gathered}$$

and hence

$$\begin{gathered}
\sup_{n\geq n_1} F(k,x_n(k))\leq\sup_{|t|\geq 1} F(k,t);\\
\dots\\
\sup_{n\geq n_l} F(k,x_n(k))\leq\sup_{|t|\geq l} F(k,t).
\end{gathered}$$

Thus, we obtain

$$\begin{aligned}
\limsup_{m\to\infty} F(k,x_m(k)) &= \lim_{m\to\infty}\sup_{n\geq m} F(k,x_n(k)) = \lim_{l\to\infty}\sup_{n\geq n_l} F(k,x_n(k))\\
&\leq \lim_{l\to\infty}\sup_{|t|\geq l} F(k,t) = \limsup_{|t|\to\infty} F(k,t), \quad (k\in\mathbb{Z}[1,T]),
\end{aligned}$$

i.e. (2.2.33). Now, passing to '$\limsup_{n\to\infty}$' in the inequality

$$\widetilde{\mathcal{E}}_{X_P}(x_n)+\sum_{k=1}^{T} F(k,x_n(k))\geq E_{X_P}(\tilde{x}_0)$$

and using (2.2.30), (2.2.32) and (2.2.33), one obtains

$$c + \sum_{k=1}^{T} \limsup_{|t|\to\infty} F(k,t) \geq E_{X_P}(\tilde{x}_0),$$

which contradicts (2.2.29). Consequently, $\{x_n\} \subset X_P$ is bounded and the proof is complete. ∎

Remark 2.2.16. (i) If in Theorem 2.2.15 we assume, in addition, that exists $c_0 \in \mathbb{R} \setminus \{0\}$ such that $\sum_{k=1}^{T} F(k, c_0) > 0$, then it is easy to see that the solution of problem (2.2.16) is nontrivial. Indeed, suppose that the solution of problem (2.2.16) obtained in Theorem 2.2.15 is the trivial solution. Then, we obtain

$$0 = \widetilde{\mathcal{E}}_{X_P}(0) \leq \widetilde{\mathcal{E}}_{X_P}(c_0) = -\sum_{k=1}^{T} F(k, c_0) < 0,$$

a contradiction.

(ii) Theorem 3.8 proved in [84] for $p =$ constant is a simple consequence of Theorem 2.2.15 (see also [18, Remark 3.4 (i)]).

(iii) From Theorem 2.1.5 we have that if (2.2.3) holds true and

$$\limsup_{|t|\to\infty} \frac{F(k,t)}{|t|^{p(k)}} < 0, \quad (\forall)\ k \in \mathbb{Z}[1,T],$$

then problem (2.2.1) has at least one solution which minimizes $\mathcal{E}_{X_P}$ on X_P. Note that this also can be inferred from Theorem 2.2.15 with $\ell = 0$.

Example 2.2.17. [see Example 3.9 in [84] for $p =$ constant] Consider the problem

$$\begin{cases} -\Delta_{p(0)} x(0) = -2x(1)\exp(-x(1)^2) + 2, \\ -\Delta_{p(1)} x(1) = -2x(2)\exp(-x(2)^2), \\ -\Delta_{p(2)} x(2) = -2x(3)\exp(-x(3)^2) - 2, \\ x(0) - x(4) = 0 = \Delta x(0) - \Delta x(3). \end{cases} \tag{2.2.34}$$

We apply Theorem 2.2.15 with $T = 3$, $p(0) = p(3)$ and $\ell(1) = 2$, $\ell(2) = 0$, $\ell(3) = -2$. One has $\tilde{x}_0(0) = \tilde{x}_0(4) = 0$, $\tilde{x}_0(1) = 1$, $\tilde{x}_0(2) = 0$, $\tilde{x}_0(3) = -1$ and $F(k,x) = \exp(-x^2) - 1$ for all $k \in \mathbb{Z}[1,3]$. Then

$$\lim_{|x|\to\infty} \sum_{k=1}^{3} F(k,x) = -3 < -2 + 2\exp(-1) = \sum_{k=1}^{3} F(k, \tilde{x}_0(k)),$$

showing that (2.2.34) has at least one solution.

Remark 2.2.18. Note that none of the Theorems 2.2.7 or 2.2.9 in the previous paragraph can be applied to problem (2.2.34).

We reformulate for the Neumann problems (2.2.14) and (2.2.26), the existence results proved above in the periodic case. These are obtained exactly as the corresponding ones for the periodic problems by no longer than 'mutatis mutandis' arguments.

Proposition 2.2.19. *If $\ell \in \mathcal{L}$, then problem* (2.2.14) *has a unique solution $\tilde{x}_0 \in X_{N,0}$. Moreover, $\tilde{x}_0$ is the unique minimizer of E_{X_N} on $X_{N,0}$.*

Theorem 2.2.20. *Let $\ell \in \mathcal{L}$ and $\tilde{x}_0 \in X_{N,0}$ be the unique solution of problem* (2.2.14). *Assume that there exists $M \geq 0$ such that* (2.2.27) *and* (2.2.28) *hold true. Then problem* (2.2.26) *given in Theorem 2.2.13 has at least one solution which minimizes $\widetilde{\mathcal{E}}_{X_N}$ on X_N.*

2.2.4 Convex potential

Under the assumption (2.2.18), the boundedness condition on f in Theorem 2.2.9 can be removed when F is concave with respect to the second variable. Notice that in this situation the potential $\mathcal{E}_{X_P}$ is convex.

Theorem 2.2.21. *Assume that* (2.2.3) *and* (2.2.18) *hold true. If $F(k,\cdot)$ is concave for all $k \in \mathbb{Z}[1,T]$, then problem* (2.2.1) *has at least one solution which minimizes $\mathcal{E}_{X_P}$ on X_P.*

Proof. By assumption (2.2.18), the function

$$t \to \sum_{k=1}^{T} F(k,t)$$

has a maximum at some point $c \in \mathbb{R}$ for which

$$\sum_{k=1}^{T} f(k,c) = 0. \tag{2.2.35}$$

We show that $\mathcal{E}_{X_P}$ admits a bounded minimizing sequence, which will conclude the proof. So, let $\{x_n\} \subset X_P$ be a minimizing sequence for $\mathcal{E}_{X_P}$. By the convexity of $-F(k,\cdot)$, for each $y,z \in \mathbb{R}$ and each $\lambda \in (0,1)$, one has

$$-\frac{F(k,z+\lambda(y-z)) - F(k,z)}{\lambda} \leq -F(k,y) + F(k,z).$$

Letting $\lambda \to 0$, we obtain

$$-F(k,y) \geq -F(k,z) - f(k,z)(y-z), \quad (\forall)\ y,z \in \mathbb{R}. \tag{2.2.36}$$

From (2.2.35) and (2.2.36), it follows that

$$\begin{aligned}\mathcal{E}_{X_P}(x_n) &\geq \varphi_{X_P}(x_n) - \sum_{k=1}^{T} F(k,c) - \sum_{k=1}^{T} f(k,c)(x_n(k)-c)\\ &= \varphi_{X_P}(x_n) - \sum_{k=1}^{T} F(k,c) - \sum_{k=1}^{T} f(k,c)\tilde{x}_n(k),\end{aligned}$$

where $x_n = \overline{\overline{x}}_n + \tilde{x}_n$ with $\overline{\overline{x}}_n = (1/T)\sum_{k=1}^{T} x_n(k)$. By the equivalence of the norms on $X_{P,0}$ and (2.2.4), we get

$$\begin{aligned}\mathcal{E}_{X_P}(x_n) &\geq \varphi_{X_P}(\tilde{x}_n) - \sum_{k=1}^{T} F(k,c) - \sum_{k=1}^{T} |f(k,c)||\tilde{x}_n(k)|\\ &\geq \|\tilde{x}_n\|_{p(\cdot)}^{p^-} - C_1\|\tilde{x}_n\|_{p(\cdot)} - C_2,\ \text{for } \|\tilde{x}_n\|_{p(\cdot)} > 1,\end{aligned}$$

where C_1 and C_2 are positive constants. Since $\{\mathcal{E}_{X_P}(x_n)\}$ is bounded, we can find some $M > 0$ such that

$$M \geq \mathcal{E}_{X_P}(x_n) \geq \|\tilde{x}_n\|_{p(\cdot)}^{p^-} - C_1\|\tilde{x}_n\|_{p(\cdot)} - C_2,\ \text{for } \|\tilde{x}_n\|_{p(\cdot)} > 1,$$

showing that $\{\|\tilde{x}_n\|_{p(\cdot)}\}$ is bounded. Next, by convexity,

$$\begin{aligned}-F\left(k, \overline{\overline{x}}_n/2\right) &= -F\left(k, (1/2)(x_n(k) - \tilde{x}_n(k))\right)\\ &\leq -\frac{1}{2}F(k, x_n(k)) - \frac{1}{2}F(k, -\tilde{x}_n(k)),\end{aligned}$$

for all $k \in \mathbb{Z}[1,T]$ and $n \in \mathbb{N}$. This, together with (2.2.4), implies

$$M \geq \mathcal{E}_{X_P}(x_n) \geq \|\tilde{x}_n\|_{p(\cdot)}^{p^-} - 2\sum_{k=1}^{T} F(k, \overline{\overline{x}}_n/2) + \sum_{k=1}^{T} F(k, -\tilde{x}_n(k)),$$

for $\|\tilde{x}_n\|_{p(\cdot)} > 1$. Thus, by the boundedness of the sequences $\{\|\tilde{x}_n\|_{p(\cdot)}\}$ and $\{\|\tilde{x}_n\|_\infty\}$, we get

$$M \geq \mathcal{E}_{X_P}(x_n) \geq -2\sum_{k=1}^{T} F(k, \overline{\overline{x}}_n/2) - C_3,$$

with $C_3 > 0$ and hence,

$$\sum_{k=1}^{T} F(k, \overline{\overline{x}}_n/2) \geq -\frac{M + C_3}{2}, \quad (\forall)\ n \in \mathbb{N}.$$

Therefore, $\sum_{k=1}^{T} F(k, \overline{\overline{x}}_n/2)$ is bounded from below and then, arguing by contradiction, from (2.2.18) easily follows that $\{\overline{\overline{x}}_n\}$ is bounded. Since $\|x_n\|_{X_P} = |\overline{\overline{x}}_n| + \|\tilde{x}_n\|_{p(\cdot)}$, we infer that $\{x_n\}$ is bounded in X_P. ■

Theorem 2.2.22. *Assume that* (2.2.3) *holds true. If, for all* $k \in \mathbb{Z}[1, T]$, $F(k, \cdot)$ *is strictly concave, then the following conditions are equivalent:*

(i) Problem (2.2.1) *is solvable;*

(ii) There exists $c \in \mathbb{R}$ *such that* $\sum_{k=1}^{T} f(k, c) = 0$*;*

(iii) $\sum_{k=1}^{T} F(k, t) \to -\infty$ *when* $|t| \to \infty$.

Proof. Assume (i) and let $x \in X_P$ be a solution of problem (2.2.1). Then, summing from 1 to T in the equation, using (2.2.3) and the boundary conditions, one obtains

$$\sum_{k=1}^{T} f(k, x(k)) = 0. \tag{2.2.37}$$

Let $x = \overline{\overline{x}} + \tilde{x}$, where $\overline{\overline{x}} = (1/T)\sum_{k=1}^{T} x(k)$ and define the strictly convex functions $\mathcal{G}$ and $\tilde{\mathcal{G}}$ on $\mathbb{R}$ by

$$\mathcal{G}(t) = -\sum_{k=1}^{T} F(k, t), \quad (\forall)\ t \in \mathbb{R},$$

respectively,

$$\tilde{\mathcal{G}}(t) = -\sum_{k=1}^{T} F(k, \tilde{x}(k) + t), \quad (\forall)\ t \in \mathbb{R}.$$

From (2.2.37), we have that $\tilde{\mathcal{G}}'(\overline{\overline{x}}) = 0$, which by Proposition 1.4, implies

$$\tilde{\mathcal{G}}(t) \to +\infty, \quad \text{as } |t| \to \infty. \tag{2.2.38}$$

Also, by the strict convexity of $-F(k, \cdot)$, one obtains

$$\tilde{\mathcal{G}}(t) < -\frac{1}{2}\sum_{k=1}^{T} F(k, 2t) - \frac{1}{2}\sum_{k=1}^{T} F(k, 2\tilde{x}(k)) = \frac{1}{2}\mathcal{G}(2t) + C(\tilde{x})$$

and by virtue of (2.2.38) it follows that $\mathcal{G}(t) \to +\infty$ as $|t| \to \infty$. Hence, from Proposition 1.4, there exists $c \in \mathbb{R}$ such that $\sum_{k=1}^{T} f(k, c) = 0$, i.e. (ii). Again, from Proposition 1.4 it results that (ii) implies (iii).

Now, on account of Theorem 2.2.21 we have that (iii) implies (i), which conclude the proof. ■

Analogously, for the Neumann problem (2.2.2), one has

Theorem 2.2.23. *If condition* (2.2.18) *holds true and $F(k,\cdot)$ is concave for all $k \in \mathbb{Z}[1,T]$, then problem* (2.2.2) *has at least one solution which minimizes $\mathcal{E}_{X_N}$ on X_N.*

Theorem 2.2.24. *If $F(k,\cdot)$ is strictly concave, for all $k \in \mathbb{Z}[1,T]$, then the following conditions are equivalent:*

(i) Problem (2.2.2) *is solvable;*

(ii) There exists $c \in \mathbb{R}$ such that $\sum_{k=1}^{T} f(k,c) = 0$;

(iii) $\sum_{k=1}^{T} F(k,t) \to -\infty$ when $|t| \to \infty$.

Remark 2.2.25. We point that Theorem 2.2.21 and Theorem 2.2.22 are discrete $p(\cdot)$-Laplacian variants of Theorem 1.8 and Theorem 1.9 in [105].

2.2.5 Lower and upper solutions for problems (2.2.1) and (2.2.2)

In the sequel we use Theorems 2.2.15 and 2.2.20 to derive the solvability of problems (2.2.1), respectively (2.2.2) by the method of lower and upper solutions. Below, if $x, y \in X_P$ (or $\in X_N$) are such that $x(k) \leq y(k)$ (resp. $x(k) < y(k)$) for all $k \in \mathbb{Z}[0,T+1]$, we shall write $x \leq y$ (resp. $x < y$).

Definition 2.2.26. A function $\alpha \in X_P$ (resp. $\beta \in X_P$) is called a *lower solution* (resp. *upper solution*) for problem (2.2.1) if

$$\Delta\alpha(0) \geq \Delta\alpha(T) \quad (\text{resp. } \Delta\beta(0) \leq \Delta\beta(T))$$

and the inequality

$$-\Delta_{p(k-1)}\alpha(k-1) \leq f(k,\alpha(k)), \ \big(\text{resp. } -\Delta_{p(k-1)}\beta(k-1) \geq f(k,\beta(k))\big) \tag{2.2.39}$$

holds for all $k \in \mathbb{Z}[1,T]$. Such a lower or upper solution will be called *strict* if the inequalities in (2.2.39) are strict.

We need the following simple consequence of Theorem 2.2.15.

Proposition 2.2.27. *Let $g : \mathbb{Z}[1,T] \times \mathbb{R} \to \mathbb{R}$ be continuous and bounded. If* (2.2.3) *holds true and $a \in \mathbb{R}^{\mathbb{Z}[1,T]}$ is such that $a(k) > 0$ ($k \in \mathbb{Z}[1,T]$), then* *problem*

$$\begin{cases} -\Delta_{p(k-1)}x(k-1)+a(k)x(k)=g(k,x(k)), \quad (\forall)\ k\in\mathbb{Z}[1,T], \\ \\ x(0)-x(T+1)=0=\Delta x(0)-\Delta x(T) \end{cases}$$

has at least one solution.

Proof. Theorem 2.2.15 applies with $\ell=0$ and $f(k,t)=g(k,t)-a(k)t$, for all $k\in\mathbb{Z}[1,T]$ and $t\in\mathbb{R}$. One has $\sum_{k=1}^{T}\limsup_{|t|\to\infty}F(k,t)=-\infty$. ■

In the next theorem, we obtain the existence of a solution for problem (2.2.1) lying between the lower and the upper solution of (2.2.1), when these are well ordered.

Theorem 2.2.28. *If* (2.2.3) *holds and problem* (2.2.1) *has a lower solution α and an upper solution β such that $\alpha\le\beta$, then* (2.2.1) *has a solution x with $\alpha\le x\le\beta$. Moreover, if α and β are strict, then $\alpha<x<\beta$.*

Proof. Let $\gamma:\mathbb{Z}[1,T]\times\mathbb{R}\to\mathbb{R}$ be the continuous function defined by

$$\gamma(k,t)=\begin{cases}\beta(k), & t>\beta(k),\\ t, & \alpha(k)\le t\le\beta(k),\\ \alpha(k), & t<\alpha(k)\end{cases} \tag{2.2.40}$$

and consider the modified problem

$$\begin{cases} -\Delta_{p(k-1)}x(k-1)+x(k)=f(k,\gamma(k,x(k)))+\gamma(k,x(k)), (\forall)\ k\in\mathbb{Z}[1,T], \\ \\ x(0)-x(T+1)=0=\Delta x(0)-\Delta x(T). \end{cases} \tag{2.2.41}$$

Using Proposition 2.2.27 with $a\equiv 1$ and $g(k,t)=f(k,\gamma(k,t))+\gamma(k,t)$, for all $k\in\mathbb{Z}[1,T]$ and $t\in\mathbb{R}$, problem (2.2.41) has at least one solution $x\in X_P$.

Then, following the same arguments as in Step I from the proof of Theorem 1 in [25] (also see [21, Theorem 1]), we show that if x is a solution of (2.2.41), then $\alpha\le x\le\beta$ and hence x in fact solves problem (2.2.1).

Suppose by contradiction that there is some $k_0\in\mathbb{Z}[0,T+1]$ such that $\alpha(k_0)-x(k_0)>0$ so that $\alpha(m)-x(m)=\max_{k\in\mathbb{Z}[0,T+1]}(\alpha(k)-x(k))>0$. Since x is solution of (2.2.41) and $\alpha\in X_P$, we have that $\alpha(0)-x(0)=\alpha(T+1)-x(T+1)$ and $\Delta(\alpha-x)(0)\ge\Delta(\alpha-x)(T)$. Hence we obtain that $m\in\mathbb{Z}[1,T]$, because, if $\alpha(k)-x(k)<\alpha(0)-x(0)=\alpha(T+1)-x(T+1)$, for all $k\in\mathbb{Z}[1,T]$, then

$$0>\alpha(1)-x(1)-(\alpha(0)-x(0))\ge\alpha(T+1)-x(T+1)-(\alpha(T)-x(T))>0,$$

a contradiction. It follows that $\alpha(m+1)-\alpha(m) \le x(m+1)-x(m)$ and $\alpha(m)-\alpha(m-1) \ge x(m)-x(m-1)$, i.e.,

$$\Delta\alpha(m) \le \Delta x(m) \quad \text{and} \quad \Delta\alpha(m-1) \ge \Delta x(m-1).$$

Taking into account (2.1) and that $h_{p(\cdot)}$ is an increasing homeomorphism, we infer that

$$\Delta_{p(m-1)}\alpha(m-1) \le \Delta_{p(m-1)}x(m-1)$$

and by virtue of (2.2.39), one obtains

$$\begin{aligned}\Delta_{p(m-1)}\alpha(m-1) \le \Delta_{p(m-1)}x(m-1)& \\ &= -f(m,\gamma(m,x(m))) - \gamma(m,x(m)) + x(m) \\ &= -f(m,\alpha(m)) + (x(m)-\alpha(m)) \\ &< -f(m,\alpha(m)) \le \Delta_{p(m-1)}\alpha(m-1),\end{aligned}$$

a contradiction. Similarly, it can be shown that $x \le \beta$. If α, β are strict, the fact that $\alpha < x < \beta$ follows as above, in a standard way. ∎

Remark 2.2.29. Theorem 2.2.28 recovers Corollary 4.3 proved in [84] for the case when p is constant.

Example 2.2.30. Let $f_1 : \mathbb{R} \to \mathbb{R}$ be a continuous function. Assume that (2.2.3) holds true and that there exist α, $\beta \in \mathbb{R}$ such that $\alpha \le \beta$ and $f_1(\alpha) \ge 0 \ge f_1(\beta)$, then the following problem

$$\begin{cases} -\Delta_{p(k-1)}x(k-1) = f_1(x(k)), \quad (\forall)\ k \in \mathbb{Z}[1,T], \\ \\ x(0)-x(T+1) = 0 = \Delta x(0) - \Delta x(T) \end{cases}$$

has at least one solution x with $\alpha \le x \le \beta$.

Analogously, for the Neumann problem the concepts of a lower and an upper solution are defined by the following

Definition 2.2.31. A function $\alpha \in X_N$ is called a *lower solution* for problem (2.2.2) if it satisfies

$$-\Delta_{p(k-1)}\alpha(k-1) \le f(k,\alpha(k)), \quad (\forall)\ k \in \mathbb{Z}[1,T], \tag{2.2.42}$$

together with

$$\Delta\alpha(T) \le 0 \le \Delta\alpha(0).$$

In the same way, an *upper solution* is a function $\beta \in X_N$ satisfying the reversed inequalities, i.e.,

$$-\Delta_{p(k-1)}\beta(k-1) \geq f(k,\beta(k)), \quad (\forall)\ k \in \mathbb{Z}[1,T], \tag{2.2.43}$$

respectively

$$\Delta\beta(0) \leq 0 \leq \Delta\beta(T).$$

Also, a lower or an upper solution will be called *strict* if the inequalities in (2.2.42), respectively (2.2.43) are strict.

The existence of a solution for problem (2.2.2) lying between the lower and the upper solution can be treated similarly to the periodic problem (2.2.1). Hence, Proposition 2.2.27 will become:

Proposition 2.2.32. *Let $g : \mathbb{Z}[1,T] \times \mathbb{R} \to \mathbb{R}$ be continuous and bounded. If $a \in \mathbb{R}^{\mathbb{Z}[1,T]}$ is such that $a(k) > 0$ for all $k \in \mathbb{Z}[1,T]$, then the problem*

$$\begin{cases} -\Delta_{p(k-1)}x(k-1) + a(k)x(k) = g(k,x(k)), \quad (\forall)\ k \in \mathbb{Z}[1,T], \\ \\ \Delta x(0) = 0 = \Delta x(T) \end{cases}$$

has at least one solution.

Theorem 2.2.33. *If problem* (2.2.2) *has a lower solution α and an upper solution β such that $\alpha \leq \beta$, then it has a solution x with $\alpha \leq x \leq \beta$. Moreover, if α and β are strict, then $\alpha < x < \beta$.*

Proof. We consider the following problem

$$\begin{cases} -\Delta_{p(k-1)}x(k-1) + x(k) = f(k,\gamma(k,x(k))) + \gamma(k,x(k)), (\forall)\ k \in \mathbb{Z}[1,T], \\ \\ \Delta x(0) = 0 = \Delta x(T), \end{cases} \tag{2.2.44}$$

with γ defined in (2.2.40). Similar to the periodic case, by Proposition 2.2.32, problem (2.2.44) has at least one solution $x \in X_N$. It remains to show that $\alpha \leq x \leq \beta$ and hence x will solve (2.2.2). Suppose by contradiction that there is some $k_0 \in \mathbb{Z}[0,T+1]$ such that $\alpha(k_0) - x(k_0) > 0$ so that $\alpha(m) - x(m) = \max_{k\in\mathbb{Z}[0,T+1]}(\alpha(k) - x(k)) > 0$.

We show that m can be chosen in $\mathbb{Z}[1,T]$. Assuming that $\alpha(k) - x(k) < \alpha(0) - x(0)$ for all $k \in \mathbb{Z}[1,T+1]$, we obtain

$$0 > \alpha(1) - x(1) - (\alpha(0) - x(0)) = \Delta\alpha(0) - \Delta x(0) = \Delta\alpha(0) \geq 0,$$

a contradiction. It follows that

$$\max_{k\in\mathbb{Z}[0,T+1]}(\alpha(k)-x(k)) = \max_{k\in\mathbb{Z}[1,T+1]}(\alpha(k)-x(k)).$$

Next, if $\alpha(k)-x(k) < \alpha(T+1)-x(T+1)$ for all $k \in \mathbb{Z}[1,T]$, then

$$0 > \alpha(T)-x(T)-(\alpha(T+1)-x(T+1)) = -\Delta\alpha(T)+\Delta x(T) = -\Delta\alpha(T) \geq 0,$$

a contradiction again. Therefore, it holds

$$\max_{k\in\mathbb{Z}[1,T+1]}(\alpha(k)-x(k)) = \max_{k\in\mathbb{Z}[1,T]}(\alpha(k)-x(k)).$$

Then, from

$$\alpha(m)-x(m) = \max_{k\in\mathbb{Z}[1,T]}(\alpha(k)-x(k)),$$

m can be chosen in $\mathbb{Z}[1,T]$. This implies

$$\Delta\alpha(m) \leq \Delta x(m) \quad \text{and} \quad \Delta\alpha(m-1) \geq \Delta x(m-1)$$

and using exactly the same arguments as in the proof of Theorem 2.2.28, from (2.2.42), we obtain that $\alpha \leq x$. Similarly, it can be shown that $x \leq \beta$.

Also, we remark that if α, β are strict, then $\alpha < x < \beta$ follows in a similar way. ∎

2.3 Multiplicity of solutions for periodic and Neumann problems

Let $r : \mathbb{Z}[1,T] \to (0,\infty)$ be a given function and λ be a positive parameter. In this section, setting

$$\mathcal{A}_k(x) := -\Delta_{p(k-1)}x(k-1) + r(k)h_{p(k)}(x(k)), \quad (k \in \mathbb{Z}[1,T]),$$

we first deal with multiplicity of solutions for the periodic problem

$$\begin{cases} \mathcal{A}_k(x) = \lambda f(k,x(k)), \quad (\forall)\ k \in \mathbb{Z}[1,T], \\ x(0)-x(T+1) = 0 = \Delta x(0)-\Delta x(T) \end{cases} \tag{2.3.1}$$

and for the Neumann problem

$$\begin{cases} \mathcal{A}_k(x) = \lambda f(k,x(k)), \quad (\forall)\ k \in \mathbb{Z}[1,T], \\ \Delta x(0) = 0 = \Delta x(T), \end{cases} \tag{2.3.2}$$

where $f : \mathbb{Z}[1, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

In the first paragraph we provide sufficient conditions which guarantee the existence of at least three solutions for suitable values of the parameter λ. In the second paragraph, under some additional assumptions on the nonlinearity f, we show by a mountain pass type argument that problems (2.3.1) and (2.3.2) have at least two positive solutions, for λ large enough.

In the presence of the Ambrosetti-Rabinowitz type condition (2.1.38), we study in Paragraph 2.3.3 the existence of multiple nontrivial solutions for the one-parameter periodic, respectively, Neumann problems:

$$\begin{cases} \mathcal{A}_k(x) = f(k, x(k)) + \lambda b(k) h_{q(k)}(x(k)), & (\forall)\ k \in \mathbb{Z}[1, T], \\ x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T) \end{cases} \tag{2.3.3}$$

and

$$\begin{cases} \mathcal{A}_k(x) = f(k, x(k)) + \lambda b(k) h_{q(k)}(x(k)), & (\forall)\ k \in \mathbb{Z}[1, T], \\ \Delta x(0) = 0 = \Delta x(T), \end{cases} \tag{2.3.4}$$

where $q : \mathbb{Z}[1, T] \to (1, \infty)$, $b : \mathbb{Z}[1, T] \to \mathbb{R}$ are given functions and as above, f is continuous. Recall that, when we refer to the periodic case, we must assume that the variable exponent p is T-periodic (see (2.2.3)). As in the previous section, we shall treat in detail the periodic problems (2.3.1) and (2.3.3) and we restrict ourselves to only point out the corresponding adaptations for the treatment of the Neumann problems (2.3.2) and (2.3.4).

To establish the multiplicity results from this section, we again use a variational approach. Thus, to treat the periodic problems (2.3.1) and (2.3.3), we consider the space X_P defined in Paragraph 2.2.1, while in the case of problems (2.3.2) and (2.3.4), we shall use the space X_N introduced in the same paragraph for problem (2.2.2).

As in the previous section, for convenience in notations we generically denote by X one of the spaces X_P or X_N. Here and hereafter, the space X will be endowed with the Luxemburg type norm $\|\cdot\|_{\eta,p(\cdot)}$ ($\eta > 0$) defined in Proposition 2.1.1. Also, for all $x \in X$ and any $\eta > 0$, we have the inequalities (2.1.7) and (2.1.8) stated in Proposition 2.1.2, still numbered here by (2.1.7), respectively (2.1.8). These will be invoked in the following, playing a technical role in the proofs. The corresponding open ball of center 0 and radius $\sigma > 0$ will be denoted by B_σ.

Now, let $\Phi_X : X \to \mathbb{R}$ be the functional defined as in (2.1.27), i.e.,

$$\Phi_X(x) = \sum_{k=1}^{T+1} \frac{|\Delta x(k-1)|^{p(k-1)}}{p(k-1)} + \sum_{k=1}^{T} \frac{r(k)}{p(k)} |x(k)|^{p(k)}, \quad (\forall)\ x \in X.$$

We have that $\Phi_X \in C^1(X,\mathbb{R})$ and its derivative is given in (2.1.28). The energy functional corresponding to problem (2.3.1) (resp. (2.3.2)) is

$$\Psi_X(x) = \Phi_X(x) - \lambda \mathcal{F}_X(x), \quad (\forall)\ x \in X,$$

with $\mathcal{F}_X$ from (2.2.11). By virtue of (2.1.28) and (2.2.12), one has

$$\langle \Psi_X'(x), y\rangle = \sum_{k=1}^{T+1} h_{p(k-1)}(\Delta x(k-1))\Delta y(k-1)$$

$$+\sum_{k=1}^{T} r(k)h_{p(k)}(x(k))y(k) - \lambda \sum_{k=1}^{T} f(k,x(k))y(k), \quad (\forall)\ x,y \in X.$$

In the case of problem (2.3.3) (resp. (2.3.4)), instead of Φ_X, one works with $\Phi_{b,X} : X \to \mathbb{R}$ defined by

$$\Phi_{b,X}(x) = \sum_{k=1}^{T+1} \frac{|\Delta x(k-1)|^{p(k-1)}}{p(k-1)} + \sum_{k=1}^{T} \frac{r(k)}{p(k)}|x(k)|^{p(k)} - \lambda \sum_{k=1}^{T} \frac{b(k)}{q(k)}|x(k)|^{q(k)},$$

for all $x \in X$. It is a simple matter to check that $\Phi_{b,X} \in C^1(X,\mathbb{R})$ and

$$\langle \Phi_{b,X}'(x), y\rangle = \sum_{k=1}^{T+1} h_{p(k-1)}(\Delta x(k-1))\Delta y(k-1)$$

$$+\sum_{k=1}^{T} r(k)h_{p(k)}(x(k))y(k) - \lambda \sum_{k=1}^{T} b(k)h_{q(k)}(x(k))y(k), \quad (\forall)\ x \in X. \quad (2.3.5)$$

The Euler-Lagrange functional associated to problem (2.3.3) (resp. (2.3.4)) will be

$$\Psi_{b,X}(x) = \Phi_{b,X}(x) - \mathcal{F}_X(x), \quad (\forall)\ x \in X,$$

with $X = X_P$ (resp. $X = X_N$). Therefore, $\Psi_{b,X} \in C^1(X,\mathbb{R})$ and from (2.3.5) and (2.2.12), we have

$$\langle \Psi_{b,X}'(x), y\rangle = \sum_{k=1}^{T+1} h_{p(k-1)}(\Delta x(k-1))\Delta y(k-1) + \sum_{k=1}^{T} r(k)h_{p(k)}(x(k))y(k)$$

$$-\lambda \sum_{k=1}^{T} b(k)h_{q(k)}(x(k))y(k) - \sum_{k=1}^{T} f(k,x(k))y(k), \quad (\forall)\ x,y \in X. \quad (2.3.6)$$

Proposition 2.3.1 (see Proposition 2.2.1)**.** *Assume that* (2.2.3) *holds true. A function $x \in X_P$ is solution of problem* (2.3.1) (*resp.* (2.3.3)) *if and only if it is a critical point of Ψ_{X_P}* (*resp.* Ψ_{b,X_P})*.*

Proposition 2.3.2 (see Proposition 2.2.3). *A function $x \in X_N$ is solution of problem* (2.3.2) (*resp.* (2.3.4)) *if and only if it is a critical point of* Ψ_{X_N} (*resp.* Ψ_{b,X_N}).

The rest of the section is organized as follows. In Paragraph 2.3.1, for each λ belonging to some well-defined intervals, we obtain the existence of at least three solutions for problems (2.3.1) and (2.3.2). For λ sufficiently large, we establish in Paragraph 2.3.2 the existence of at least two positive solutions for problems (2.3.1) and (2.3.2). Paragraph 2.3.3 is devoted to problems (2.3.3) and (2.3.4); we obtain the existence of at least two non-trivial solutions, if λ is small enough.

The results in Paragraph 2.3.1 are proved in [34] and the ones from Paragraphs 2.3.2 and 2.3.3 are obtained in [123].

2.3.1 Three solutions for problems (2.3.1) and (2.3.2)

The first results ensure the existence of an open interval Λ_h, such that for every $\lambda \in \Lambda_h$, problems (2.3.1) and (2.3.2) admit at least three solutions whose norms are bounded uniformly with respect to λ. Moreover, an upper bound for Λ_h is established.

Hereafter, for each positive constant c, we shall use the notations:

$$\Gamma_{min}(c) := \frac{\sum_{k=1}^{T} F(k,c)}{\min\{c^{p^-}, c^{p^+}\}} \quad \text{and} \quad \Gamma_{max}(c) := \frac{\sum_{k=1}^{T} F(k,c)}{\max\{c^{p^-}, c^{p^+}\}}.$$

Theorem 2.3.3. *We assume that there exist positive constants c, d with $c < d$ such that*

$$\sum_{k=1}^{T} \sup_{|t|<c} F(k,t) < \frac{p^- \underline{r} \min\{c^{p^-}, c^{p^+}\}}{p^+ \sum_{k=1}^{T} r(k)} \, \Gamma_{max}(d) \tag{2.3.7}$$

and

$$\limsup_{|t|\to\infty} \frac{F(k,t)}{|t|^{p(k)}} \le 0, \quad (\forall)\ k \in \mathbb{Z}[1,T]. \tag{2.3.8}$$

Also, we set

$$\tilde{a} = \left(\frac{p^- \Gamma_{max}(d)}{\sum_{k=1}^{T} r(k)} - \frac{p^+ \sum_{k=1}^{T} \sup_{|t|<c} F(k,t)}{\underline{r} \min\{c^{p^-}, c^{p^+}\}} \right)^{-1}. \tag{2.3.9}$$

If (2.2.3) *holds true, then for every $h > 1$, there exists an open interval $\Lambda_h \subseteq [0, h\tilde{a}]$ and a positive real number μ such that, for all $\lambda \in \Lambda_h$, problem* (2.3.1) *admits at least three solutions in X_P, whose norms are less than μ.*

Proof. We apply Theorem 1.17 with $Y = X_P$, $\psi = \Phi_{X_P}$ and $J = \mathcal{F}_{X_P}$. It is clear that the regularity assumptions required on Φ_{X_P}, $\mathcal{F}_{X_P}$ and X_P are satisfied. Also, $\Phi_{X_P}(0) = \mathcal{F}_{X_P}(0) = 0$ and $\Phi_{X_P}(x) \geq 0$, for all $x \in X_P$.

We denote

$$\omega = \frac{\underline{r}\min\{c^{p^-}, c^{p^+}\}}{p^+} > 0$$

and since $c < d$, one has

$$\Phi_{X_P}(d) = \sum_{k=1}^{T} \frac{r(k)}{p(k)} d^{p(k)} \geq \frac{\min\{d^{p^-}, d^{p^+}\}}{p^+} \sum_{k=1}^{T} r(k) > \frac{\underline{r}\min\{c^{p^-}, c^{p^+}\}}{p^+} = \omega, \tag{2.3.10}$$

that is, condition (i_1) from Theorem 1.17, with $y_1(k) \equiv d \in X_P$, for all $k \in \mathbb{Z}[0, T+1]$. Also, it is easy to see that (2.3.7) implies

$$\sum_{k=1}^{T} F(k, d) > 0. \tag{2.3.11}$$

Now, if $\Phi_{X_P}(x) < \omega$, then

$$\sum_{k=1}^{T} |x(k)|^{p(k)} < \min\{c^{p^-}, c^{p^+}\}$$

and hence, for each $k \in \mathbb{Z}[1, T]$, one obtains

$$|x(k)| < \min\{c^{p^-}, c^{p^+}\}^{\frac{1}{p(k)}}.$$

If $c \geq 1$, then $|x(k)| < c^{\frac{p^-}{p(k)}} < c$. Also, $|x(k)| < c^{\frac{p^+}{p(k)}} < c$, provided that $c \in (0, 1)$. Therefore, $\max_{k\in\mathbb{Z}[1,T]} |x(k)| < c$ and from (2.3.7) and (2.3.11), we infer

$$\begin{aligned}
\sup_{\Phi_{X_P}(x)<\omega} \mathcal{F}_{X_P}(x) &\leq \sup_{\max_{k\in\mathbb{Z}[1,T]} |x(k)|<c} \mathcal{F}_{X_P}(x) \leq \sum_{k=1}^{T} \sup_{|t|<c} F(k, t) \\
&< \frac{p^-\underline{r}\min\{c^{p^-}, c^{p^+}\}}{p^+ \sum_{k=1}^{T} r(k)} \, \Gamma_{max}(d) \\
&= \frac{\omega p^- \mathcal{F}_{X_P}(d)}{\max\{d^{p^-}, d^{p^+}\} \sum_{k=1}^{T} r(k)} \leq \omega \frac{\mathcal{F}_{X_P}(d)}{\Phi_{X_P}(d)}.
\end{aligned}$$

Thus, condition (i_2) in Theorem 1.17 is satisfied. Moreover, we have

$$\overline{a} = \frac{h\omega}{\omega\frac{\mathcal{F}_{X_P}(d)}{\Phi_{X_P}(d)} - \sup_{\Phi_{X_P}(x)<\omega} \mathcal{F}_{X_P}(x)}$$

$$\leq \frac{h\omega}{\frac{p^-\underline{r}\min\{c^{p^-},c^{p^+}\}\Gamma_{max}(d)}{p^+\sum_{k=1}^T r(k)} - \sum_{k=1}^T \sup_{|t|<c} F(k,t)} = h\tilde{a},$$

with $\tilde{a}$ given in (2.3.9). Now, we consider $g(k,t) = \lambda f(k,t) - r(k)h_{p(k)}(t)$, for all $k \in \mathbb{Z}[1,T]$ and $t \in \mathbb{R}$. Then,

$$G(k,t) = \int_0^t g(k,\tau)d\tau = \lambda F(k,t) - r(k)\frac{|t|^{p(k)}}{p(k)}$$

and in view of (2.3.8), we obtain the following Hammerstein type condition

$$\limsup_{|t|\to\infty} \frac{G(k,t)}{|t|^{p(k)}} \leq -\frac{r(k)}{p(k)} < 0, \quad (\forall)\ k \in \mathbb{Z}[1,T]$$

and Theorem 2.1.5 applies choosing j the indicator function of the set $\{(x,x),\ x \in \mathbb{R}\}$. Consequently, $\Phi_{X_P} - \lambda\mathcal{F}_{X_P}$ (i.e., Ψ_{X_P}) is coercive, for every $\lambda > 0$. So, condition (i_3) in Theorem 1.17 is fulfilled. Also, it is easy to see that from coercivity, Ψ_{X_P} satisfies (PS) condition.

Thus, from Theorem 1.17, for every $h > 1$, there exists an open interval $\Lambda_h \subseteq [0, h\tilde{a}]$ and a positive real number μ such that, for all $\lambda \in \Lambda_h$, equation

$$\Phi'_{X_P}(x) - \lambda\mathcal{F}'_{X_P}(x) = 0$$

admits at least three solutions in X_P and by virtue of (2.2.3) and Proposition 2.3.1, we have that problem (2.3.1) admits at least three solutions in X_P, whose norms are less than μ and the proof is complete. ∎

Remark 2.3.4. (i) We note that according to the proof of Theorem 1.17 (also see [28, Proposition 1.3]), μ entering in Theorem 2.3.3 satisfies

$$\sup_{\Phi_{X_P}(x)<\omega} \mathcal{F}_{X_P}(x) + \frac{\omega\frac{\mathcal{F}_{X_P}(d)}{\Phi_{X_P}(d)} - \sup_{\Phi_{X_P}(x)<\omega}\mathcal{F}_{X_P}(x)}{h} < \mu < \omega\frac{\mathcal{F}_{X_P}(d)}{\Phi_{X_P}(d)},$$

for every $h > 1$.

(ii) It is worth to point out that applying Theorem 30 in [33], under the assumptions (2.2.3), (2.3.7) and (2.3.8) from Theorem 2.3.3, we also obtain in a similar way as above that, for every

$$\lambda \in \left(\frac{\sum_{k=1}^T r(k)}{p^-\Gamma_{max}(d)},\ \frac{\underline{r}\min\{c^{p^-},c^{p^+}\}}{p^+\sum_{k=1}^T \sup_{|t|<c} F(k,t)}\right),$$

problem (2.3.1) admits at least three solutions in X_P, such that at least one is in

$$\Phi_{X_P}^{-1}\left(-\infty,\ \frac{\underline{r}\min\{c^{p^-},c^{p^+}\}}{p^+}\right)$$

and another one in

$$\Phi_{X_P}^{-1}\left(\frac{\underline{r}\min\{c^{p^-},c^{p^+}\}}{p^+},\ +\infty\right).$$

(iii) Theorem 3.2 proved in [27] by L.H. Bian *et al.* for $p=$ constant is an immediate consequence of Theorem 2.3.3.

Example 2.3.5. Let $p^-=5$, $p^+=17$, $T=15$, $\lambda>0$, $r(k)\equiv 1$ and $f(k,t)=2k(t^3-t)$, for all $k\in\mathbb{Z}[1,15]$ and $t\in\mathbb{R}$. We consider the problem

$$\begin{cases} -\Delta_{p(k-1)}x(k-1)+h_{p(k)}(x(k))=2\lambda k(x(k)^3-x(k)), \quad (\forall)\ k\in\mathbb{Z}[1,15], \\ x(0)-x(16)=0=\Delta x(0)-\Delta x(15). \end{cases} \tag{2.3.12}$$

We have

$$F(k,t)=k\left(\frac{t^4}{2}-t^2\right), \quad (\forall)\ k\in\mathbb{Z}[1,15],\ (\forall)\ t\in\mathbb{R}.$$

If we choose $c=1$ and $d=2$, then it is easy to see that the conditions of Theorem 2.3.3 are satisfied. Hence, if $p(0)=p(15)$, then for every $h>1$, there exists an open interval $\Lambda_h\subseteq[0,2^{12}h/5]$ and a positive real number μ such that, for every $\lambda\in\Lambda_h$, problem (2.3.12) has at least three solutions in X_P, whose norms are less than μ.

For the Neumann problem (2.3.2), using exactly the same strategy as above we have the following multiplicity result.

Theorem 2.3.6. *Assume that there exist positive constants c,d with $c<d$ such that* (2.3.7) *and* (2.3.8) *hold true and let $\tilde{a}$ be given by* (2.3.9). *Then, for every $h>1$, there exists an open interval $\Lambda_h\subseteq[0,h\tilde{a}]$ and a positive real number μ such that, for all $\lambda\in\Lambda_h$, problem* (2.3.2) *admits at least three solutions in X_N, whose norms are less than μ.*

Next, under a suitable sign condition on the nonlinearity f and without assuming any asymptotic condition on the primitive F of f, we obtain the existence of at least three positive solutions for problems (2.3.1) and (2.3.2), for each λ belonging to a well-defined interval.

In the periodic case, we shall need the following maximum principle.

Lemma 2.3.7. *If*

$$\begin{cases} \mathcal{A}_k(x) \geq 0, \quad (\forall)\ k \in \mathbb{Z}[1,T], \\ \\ x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T), \end{cases} \tag{2.3.13}$$

then either $x > 0$ in $\mathbb{Z}[0, T+1]$ or $x \equiv 0$.

Proof. Let $j \in \mathbb{Z}[1,T]$ be such that $x(j) = \min_{k\in\mathbb{Z}[1,T]} x(k)$. Clearly,

$$\Delta x(j) \geq 0 \quad \text{and} \quad \Delta x(j-1) \leq 0. \tag{2.3.14}$$

From (2.3.13), (2.1) and (2.3.14), we obtain

$$r(j)h_{p(j)}(x(j)) \geq |\Delta x(j)|^{p(j)-2}\Delta x(j) - |\Delta x(j-1)|^{p(j-1)-2}\Delta x(j-1) \geq 0,$$

which implies that $x(j) \geq 0$ and hence $x(k) \geq 0$ for all $k \in \mathbb{Z}[1,T]$. Then, since $x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T)$, we infer that $x(k) \geq 0$ for all $k \in \mathbb{Z}[0, T+1]$. Moreover, assuming that $x(j) = 0$, from the previous inequality and nonnegativity of $x(j-1)$ and $x(j+1)$, we have

$$0 \leq |x(j+1)|^{p(j)-2}x(j+1) + |x(j-1)|^{p(j-1)-2}x(j-1) \leq 0$$

and so, $x(j+1) = x(j-1) = 0$. Thus, repeating these arguments, the conclusion follows at once. ■

Theorem 2.3.8. *Let f be a positive continuous function on $\mathbb{Z}[1,T]\times[0,\infty)$. Assume that there exist three positive constants c_1, d and c_2, with $c_1 < d$, such that*

$$\max\{d^{p^-}, d^{p^+}\} < \frac{p^-}{2p^+}\frac{\underline{r}}{\sum_{k=1}^T r(k)} \min\{c_2^{p^-}, c_2^{p^+}\} \tag{2.3.15}$$

and

$$\max\{\Gamma_{min}(c_1),\ 2\Gamma_{min}(c_2)\} < \frac{p^-}{2p^+}\frac{\underline{r}}{\sum_{k=1}^T r(k)}\ \Gamma_{max}(d). \tag{2.3.16}$$

If (2.2.3) *holds true, then for each*

$$\lambda \in \left(\frac{2\sum_{k=1}^T r(k)}{p^-\Gamma_{max}(d)}, \frac{\underline{r}}{p^+\max\{\Gamma_{min}(c_1),\ 2\Gamma_{min}(c_2)\}}\right), \tag{2.3.17}$$

problem (2.3.1) *admits at least three distinct positive solutions x_1, x_2, x_3 in X_P, such that*

$$x_i(k) < c_2, \quad (\forall)\ k \in \mathbb{Z}[0, T+1],\ i = 1,2,3.$$

Proof. Without loss of generality, we may assume that $f(k,t) = f(k,0)$, for all $(k,t) \in \mathbb{Z}[1,T] \times (-\infty, 0)$. We shall apply Theorem 1.18 with $Y = X_P$, $\psi = \Phi_{X_P}$ and $J = \mathcal{F}_{X_P}$. From (2.1.7), we have

$$\Phi_{X_P}(x) \geq \|x\|_{\underline{r},p(\cdot)}^{p^-}, \quad (\forall)\ x \in X_P,\ \|x\|_{\underline{r},p(\cdot)} > 1,$$

which imply that Φ_{X_P} is coercive. Clearly, Theorem 1.18 (i_4) is fulfilled.

Let u_1, u_2 be two local minima of Ψ_{X_P}. They are two solutions of problem (2.3.1) and owing to Lemma 2.3.7, one has $\xi u_1(k)+(1-\xi)u_2(k) \geq 0$, for all $k \in \mathbb{Z}[0, T+1]$ and all $\xi \in [0,1]$. Hence,

$$\mathcal{F}_{X_P}(\xi u_1 + (1-\xi)u_2) \geq 0, \quad (\forall)\ \xi \in [0,1]$$

and (i_5) in Theorem 1.18 is verified.

Setting

$$\omega_1 = \frac{\underline{r}\min\{c_1^{p^-}, c_1^{p^+}\}}{p^+} \quad \text{and} \quad \omega_2 = \frac{\underline{r}\min\{c_2^{p^-}, c_2^{p^+}\}}{p^+},$$

in the same way as in the proof of Theorem 2.3.3, if $\Phi_{X_P}(x) < \omega_1$ (resp. $\Phi_{X_P}(x) < \omega_2$), we have that

$$\max_{k\in\mathbb{Z}[1,T]} |x(k)| < c_1 \Big(\text{resp. } \max_{k\in\mathbb{Z}[1,T]} |x(k)| < c_2\Big).$$

Therefore, one obtains

$$\begin{aligned} \frac{\sup_{x\in\Phi_{X_P}^{-1}(-\infty,\omega_1)} \mathcal{F}_{X_P}(x)}{\omega_1} &\leq \frac{\sup_{\max_{k\in\mathbb{Z}[1,T]}|x(k)|<c_1} \mathcal{F}_{X_P}(x)}{\omega_1} \\ &\leq \frac{\sum_{k=1}^T \sup_{|t|<c_1} F(k,t)}{\omega_1} \\ &\leq \frac{\sum_{k=1}^T F(k,c_1)}{\omega_1} = \frac{p^+}{\underline{r}}\ \Gamma_{min}(c_1), \end{aligned} \tag{2.3.18}$$

as well as

$$\frac{\sup_{x\in\Phi_{X_P}^{-1}(-\infty,\omega_2)} \mathcal{F}_{X_P}(x)}{\omega_2} \leq \frac{p^+}{\underline{r}}\ \Gamma_{min}(c_2). \tag{2.3.19}$$

On the other hand, since $c_1 < d$, we get $\Phi_{X_P}(d) > \omega_1$ (see (2.3.10)). Also, from (2.3.15), one has

$$\Phi_{X_P}(d) \leq \frac{\max\{d^{p^-}, d^{p^+}\}}{p^-} \sum_{k=1}^T r(k) < \frac{\underline{r}\min\{c_2^{p^-}, c_2^{p^+}\}}{2p^+} = \frac{\omega_2}{2}.$$

So, we have $\omega_1 < \Phi_{X_P}(d) < \omega_2/2$. Now, using (2.3.16), (2.3.18), (2.3.19) and the fact that $\sum_{k=1}^{T} F(k,d) > 0$, we infer

$$\frac{\sup_{x\in\Phi_{X_P}^{-1}(-\infty,\omega_1)} \mathcal{F}_{X_P}(x)}{\omega_1} \le \frac{p^+}{\underline{r}}\, \Gamma_{min}(c_1) < \frac{p^-}{2\sum_{k=1}^{T} r(k)}\, \Gamma_{max}(d) \le \frac{\mathcal{F}_{X_P}(d)}{2\Phi_{X_P}(d)},$$

respectively,

$$\frac{\sup_{x\in\Phi_{X_P}^{-1}(-\infty,\omega_2)} \mathcal{F}_{X_P}(x)}{\omega_2} \le \frac{p^+}{\underline{r}}\, \Gamma_{min}(c_2) < \frac{p^-}{4\sum_{k=1}^{T} r(k)}\, \Gamma_{max}(d) \le \frac{\mathcal{F}_{X_P}(d)}{4\Phi_{X_P}(d)}$$

and condition (i_6) in Theorem 1.18 holds true, with $v(k) \equiv d \in X_P$ for all $k \in \mathbb{Z}[0, T+1]$. Further, again from (2.3.18) and (2.3.19), one has that

$$\lambda \in \left(\frac{2\Phi_{X_P}(d)}{\mathcal{F}_{X_P}(d)}, \min\left\{ \frac{\omega_1}{\sup_{x\in\Phi_{X_P}^{-1}(-\infty,\omega_1)} \mathcal{F}_{X_P}(x)}, \frac{\omega_2/2}{\sup_{x\in\Phi_{X_P}^{-1}(-\infty,\omega_2)} \mathcal{F}_{X_P}(x)} \right\} \right).$$

Therefore, the functional Ψ_{X_P} admits at least three critical points $x_i \in X_P$, $i = 1, 2, 3$, which on account of (2.2.3) and Proposition 2.3.1 are solutions of problem (2.3.1) and owing to Lemma 2.3.7, are positive functions.

Finally, for $i = 1, 2, 3$, since $\omega_1 < \omega_2$ (result from $c_1 < d$ and (2.3.15)) and $\Phi_{X_P}(x_i) < \omega_2$, we get that

$$\max_{k\in\mathbb{Z}[1,T]} x_i(k) < c_2$$

and the end points inequality follows from the boundary conditions and the proof is complete. ■

Remark 2.3.9. Since the range of solutions obtained in Theorem 2.3.8 is in $[0, c_2]$, the conclusion still remains true if we assume that f is a positive continuous function only in $\mathbb{Z}[1,T] \times [0, c_2]$. Indeed, it suffices to apply this theorem to the function $\tilde{f}$ defined as $\tilde{f}(k,t) = f(k,t)$ if $t \le c_2$ and $\tilde{f}(k,t) = f(k, c_2)$ if $t \ge c_2$, for every $k \in \mathbb{Z}[1,T]$.

Example 2.3.10. Let $p^- = 7$, $p^+ = 12$, $T = 15$, $\lambda > 0$, $r(k) \equiv 1$ and

$$f(k,t) = \begin{cases} k, & 0 \le t < 1, \\ kt^{18}, & 1 \le t < 5, \\ k(10-t)^{18}, & 5 \le t < 9, \\ k, & t \ge 9, \end{cases}$$

for all $k \in \mathbb{Z}[1, 15]$, $t \in [0, \infty)$. By a simple computation we see that the conditions in Theorem 2.3.8 are satisfied if we choose $c_1 = 1$, $d = 2$ and $c_2 = 9$. Hence, if $p(0) = p(15)$, then for any $\lambda \in (19/3584, 19/1440)$, the problem

$$\begin{cases} -\Delta_{p(k-1)}x(k-1) + h_{p(k)}(x(k)) = \lambda f(k,t), \quad (\forall)\ k \in \mathbb{Z}[1,15], \\ \\ x(0) - x(16) = 0 = \Delta x(0) - \Delta x(15) \end{cases}$$

has at least three distinct positive solutions $x_i \in X_P,\ i = 1,2,3$, such that, for each $k \in \mathbb{Z}[0,16]$, one has $x_i < 9,\ i = 1,2,3$.

It is a simple matter to see that Lemma 2.3.7 remains valid with

$$\begin{cases} \mathcal{A}_k(x) \geq 0, \quad (\forall)\ k \in \mathbb{Z}[1,T], \\ \Delta x(0) = 0 = \Delta x(T), \end{cases}$$

instead of (2.3.13). Then, for the Neumann problem (2.3.2), by no longer than 'mutatis mutandis' arguments (also see Remark 2.3.9), we have

Theorem 2.3.11. *Assume that there exist three positive constants* c_1, d *and* c_2, *with* $c_1 < d$, *such that* (2.3.15) *and* (2.3.16) *hold true. If* f *is a positive continuous function on* $\mathbb{Z}[1,T]\times[0,c_2]$, *then for each* λ *as in* (2.3.17), *problem* (2.3.2) *admits at least three distinct positive solutions* $x_1,\ x_2,\ x_3$ *in* X_N, *such that*

$$x_i(k) < c_2, \quad (\forall)\ k \in \mathbb{Z}[0,T+1],\ i = 1,2,3.$$

2.3.2 Two positive solutions for problems (2.3.1) and (2.3.2)

Here, we deal with the existence of two positive solutions for problems (2.3.1) and (2.3.2), for sufficiently large values of the parameter λ. The main tools in obtaining such a result will be Theorem 1.6 and Theorem 1.7.

Theorem 2.3.12. *Assume* (2.2.3) *and that there is some* $\xi > 0$ *such that* $f(k,\cdot) > 0$ *on* $(0,\xi)$ *and* $f(k,\xi) = 0$, *for all* $k \in \mathbb{Z}[1,T]$. *If*

$$\lim_{t\to 0} \frac{f(k,t)}{|t|^{p(k)-1}} = 0, \quad (\forall)\ k \in \mathbb{Z}[1,T], \tag{2.3.20}$$

then there exists $\lambda^\star > 0$ *such that, for all* $\lambda > \lambda^\star$, *problem* (2.3.1) *has at least two positive solutions.*

Proof. We set $\tilde{f}(k,t) = f(k,t)$ for $t \in [0,\xi]$ and $\tilde{f}(k,t) = 0$ otherwise ($k \in \mathbb{Z}[1,T]$). Let us consider the problem

$$\begin{cases} \mathcal{A}_k(x) = \lambda \tilde{f}(k,x(k)), \quad (\forall)\ k \in \mathbb{Z}[1,T], \\ x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T). \end{cases} \tag{2.3.21}$$

Its energy functional is

$$\widetilde{\Psi}_{X_P}(x) = \Phi_{X_P}(x) - \lambda \sum_{k=1}^{T} \widetilde{F}(k, x(k)), \quad (\forall)\ x \in X_P,$$

where $\widetilde{F}$ is the primitive of $\tilde{f}$ (see (2.1.13)). Clearly, $\widetilde{\Psi}_{X_P}(0) = 0$.

For $x \in X_P$, setting $x_+ := \max\{x, 0\}$, we have that

$$|\Delta x(k)||\Delta x_+(k)| = \Delta x(k)\Delta x_+(k) \text{ and } |\Delta x(k)| \geq |\Delta x_+(k)|, \tag{2.3.22}$$

for all $k \in \mathbb{Z}[0, T]$. Indeed, for an arbitrary $k \in \mathbb{Z}[0, T]$, if $x(k+1)$ and $x(k)$ are of the same sign, then it is easy to see that (2.3.22) is fulfilled. Now, assume that $x(k+1) > 0 > x(k)$. Then

$$\Delta x_+(k) = x_+(k+1) - x_+(k) = x(k+1) - 0 > 0$$

and

$$\Delta x(k) = x(k+1) - x(k) > x(k+1) > 0,$$

which clearly implies (2.3.22). The same argument works if $x(k+1) < 0 < x(k)$. In this case $\Delta x_+(k) = -x(k) < 0$, respectively $\Delta x(k) < -x(k) < 0$ and again (2.3.22) holds true. From (2.3.22), we infer

$$\begin{aligned} |\Delta x_+(k)|^{p(k)} &= |\Delta x_+(k)|^{p(k)-1}|\Delta x_+(k)| \leq |\Delta x(k)|^{p(k)-1}|\Delta x_+(k)| \\ &= |\Delta x(k)|^{p(k)-2}|\Delta x(k)||\Delta x_+(k)| \\ &= |\Delta x(k)|^{p(k)-2}\Delta x(k)\Delta x_+(k), \end{aligned} \tag{2.3.23}$$

for all $k \in \mathbb{Z}[0, T]$. Now, assume that x is a nontrivial solution of problem (2.3.21). Then, using (2.3.23) and the fact that $r > 0$ on $\mathbb{Z}[1, T]$, we obtain

$$\begin{aligned} 0 = \langle \widetilde{\Psi}'_{X_P}(x), (x-\xi)_+ \rangle &= \sum_{k=1}^{T+1} h_{p(k-1)}(\Delta x(k-1))\Delta(x(k-1)-\xi)_+ \\ &+ \sum_{k=1}^{T} r(k)h_{p(k)}(x(k))(x(k)-\xi)_+ - \lambda \sum_{k=1}^{T} \tilde{f}(k, x(k))(x(k)-\xi)_+ \\ &= \sum_{k=0}^{T} |\Delta(x(k)-\xi)|^{p(k)-2}\Delta(x(k)-\xi)\Delta(x(k)-\xi)_+ \\ &+ \sum_{k=1}^{T} r(k)h_{p(k)}(x(k))(x(k)-\xi)_+ \\ &\geq \sum_{k=0}^{T} |\Delta(x(k)-\xi)_+|^{p(k)} + \sum_{k=1}^{T} r(k)|x(k)|^{p(k)-2}[(x(k)-\xi)_+]^2, \end{aligned}$$

which implies that $x(k) \leq \xi$, for all $k \in \mathbb{Z}[0, T+1]$ (also see Remark 2.3.13 below for a different proof of this fact). Since x is assumed to be a nontrivial solution for (2.3.21) and $\tilde{f}(k,t) \geq 0$ for all $k \in \mathbb{Z}[1,T]$ and $t \in \mathbb{R}$, owing to Lemma 2.3.7, we have that $x(k) > 0$ $(k \in \mathbb{Z}[0, T+1])$. Hence,

$$0 < x(k) \leq \xi \quad (\forall)\ k \in \mathbb{Z}[0, T+1]$$

and by virtue of Proposition 2.3.1, to find positive solutions of problem (2.3.1), it suffices to produce nontrivial critical points of the functional $\widetilde{\Psi}_{X_P}$.

Next, we show that $\widetilde{\Psi}_{X_P}$ satisfies the (PS) condition. With this aim, let $\{x_n\} \subset X_P$ be a sequence with $\widetilde{\Psi}_{X_P}(x_n) \leq M$, for all $n \in \mathbb{N}$. Since X_P is finite dimensional, it suffices to prove that $\{x_n\}$ is bounded. The boundedness of $\tilde{f}$ implies that there exists $C_1 > 0$ such that

$$|\widetilde{F}(k,t)| \leq C_1 |t|, \quad (\forall)\ k \in \mathbb{Z}[1,T],\ (\forall)\ t \in \mathbb{R}$$

and using the equivalence of the norms on X_P and (2.1.7), if $\|x_n\|_{\underline{r},p(\cdot)} > 1$, one has

$$M \geq \widetilde{\Psi}_{X_P}(x_n) \geq \sum_{k=1}^{T+1} \frac{|\Delta x_n(k-1)|^{p(k-1)}}{p(k-1)} + \underline{r} \sum_{k=1}^{T} \frac{|x_n(k)|^{p(k)}}{p(k)} - \lambda C_1 \sum_{k=1}^{T} |x_n(k)|$$

$$\geq \|x_n\|_{\underline{r},p(\cdot)}^{p^-} - \lambda C_1 T \|x_n\|_\infty \geq \|x_n\|_{\underline{r},p(\cdot)}^{p^-} - \lambda C_1 C_2 T \|x_n\|_{\underline{r},p(\cdot)}, \qquad (2.3.24)$$

with C_2 a positive constant. Thus, we get that $\{x_n\}$ is bounded in X_P. Also, note that (see (2.3.24))

$$\widetilde{\Psi}_{X_P}(x) \geq \|x\|_{\underline{r},p(\cdot)}^{p^-} - \lambda C_1 C_2 T \|x\|_{\underline{r},p(\cdot)}, \quad (\forall)\ x \in X_P,\ \|x\|_{\underline{r},p(\cdot)} > 1,$$

which implies that $\widetilde{\Psi}_{X_P}$ is bounded from below. Hence, by Theorem 1.7,

$$c_\lambda := \inf_{x \in X_P} \widetilde{\Psi}_{X_P}(x)$$

is a critical value of $\widetilde{\Psi}_{X_P}$, for all $\lambda > 0$.

Now, let $y \in X_P \setminus \{0\}$ such that $y(k) \in (0, \xi)$, for all $k \in \mathbb{Z}[1,T]$. As

$$\sum_{k=1}^{T} \widetilde{F}(k, y(k)) > 0,$$

we can find $\lambda^\star > 0$ sufficiently large, such that $\widetilde{\Psi}_{X_P}(y) < 0$, for all $\lambda > \lambda^\star$. For such λ, the energy functional $\widetilde{\Psi}_{X_P}$ has a critical value $c_\lambda < 0$ and a corresponding nontrivial critical point $x_{\lambda,1}$, which is a positive solution of problem (2.3.1).

Let $\lambda > \lambda^\star$ be fixed. Next, we shall produce a second nontrivial critical point of $\widetilde{\Psi}_{X_P}$ by the Mountain Pass Theorem. So, we show that $\widetilde{\Psi}_{X_P}$ has the geometry required by Theorem 1.6.

By the equivalence of the norms on X_P, for any $\eta > 0$, there is some $C_\eta > 0$ such that

$$\|x\|_\infty \leq C_\eta \|x\|_{\eta,p(\cdot)}, \quad (\forall)\ x \in X_P. \tag{2.3.25}$$

From (2.3.20), we can find constants $\sigma \in (0, \underline{r})$, $\rho \in (0, \min\{1, \|x_{\lambda,1}\|_{\sigma,p(\cdot)}/2\})$ and $C_\sigma > 0$ such that

$$\widetilde{F}(k,t) \leq \frac{\underline{r} - \sigma}{\lambda p(k)} |t|^{p(k)}, \quad (\forall)\ k \in \mathbb{Z}[1,T],\ (\forall)\ t \in \mathbb{R} \text{ with } |t| \leq \rho C_\sigma. \tag{2.3.26}$$

For $x \in X_P$ with $\|x\|_{\sigma,p(\cdot)} \leq \rho$, from (2.3.25) and (2.3.26), one has

$$\widetilde{F}(k,x(k)) \leq \frac{\underline{r} - \sigma}{\lambda p(k)} |x(k)|^{p(k)}, \quad (\forall)\ k \in \mathbb{Z}[1,T],$$

which, together with (2.1.8), imply

$$\widetilde{\Psi}_{X_P}(x) = \Phi_{X_P}(x) - \lambda \sum_{k=1}^{T} \widetilde{F}(k,x(k)) \geq \Phi_{X_P}(x) + (\sigma - \underline{r}) \sum_{k=1}^{T} \frac{1}{p(k)} |x(k)|^{p(k)}$$

$$\geq \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta x(k-1)|^{p(k-1)} + \sigma \sum_{k=1}^{T} \frac{1}{p(k)} |x(k)|^{p(k)} \geq \|x\|_{\sigma,p(\cdot)}^{p^+}$$

and Theorem 1.6 (i) is fulfilled with $\alpha = \rho^{p^+}$.

Since $\|x_{\lambda,1}\|_{\sigma,p(\cdot)} > \rho$ and $\widetilde{\Psi}_{X_P}(x_{\lambda,1}) < 0$, condition (ii) in Theorem 1.6 is also satisfied. Consequently, there exists a second nontrivial critical point $x_{\lambda,2}$ of $\widetilde{\Psi}_{X_P}$ with $\widetilde{\Psi}_{X_P}(x_{\lambda,2}) > 0$. Clearly $x_{\lambda,2}$ is distinct from $x_{\lambda,1}$. Hence, $x_{\lambda,2}$ is a second positive solution of (2.3.1) and the proof is complete. ■

Remark 2.3.13. We give here another proof of the fact that if x is a solution of problem (2.3.21), then $x(k) \leq \xi$ ($k \in \mathbb{Z}[0,T+1]$), with ξ from the assumptions in Theorem 2.3.12.

So, let $j \in \mathbb{Z}[1,T]$ be such that $x(j) = \max_{k \in \mathbb{Z}[1,T]} x(k)$. Clearly,

$$\Delta x(j) \leq 0 \quad \text{and} \quad \Delta x(j-1) \geq 0. \tag{2.3.27}$$

Suppose by contradiction that $x(j) > \xi$ (> 0). Hence, $\tilde{f}(j,x(j)) = 0$. This and the fact that x is solution of (2.3.21), yields

$$r(j) h_{p(j)}(x(j)) = |\Delta x(j)|^{p(j)-2} \Delta x(j) - |\Delta x(j-1)|^{p(j-1)-2} \Delta x(j-1)$$

and from (2.3.27), we have

$$r(j)h_{p(j)}(x(j)) \leq 0,$$

which, since $r > 0$ on $\mathbb{Z}[1,T]$, implies that $x(j) \leq 0$, a contradiction. Therefore, $x(k) \leq \xi$ for all $k \in \mathbb{Z}[1,T]$. Then, using the boundary conditions, easily follows that $x(k) \leq \xi$ for all $k \in \mathbb{Z}[0,T+1]$.

Remark 2.3.14. (i) Theorem 3.1 proved in [27] by L.H. Bian *et al.* for $p =$ constant is an immediate consequence of Theorem 2.3.12.

(ii) The idea from the proof of Theorem 2.3.12 was first used in 1973 for some elliptic partial differential equations by A. Ambrosetti and P.H. Rabinowitz [5] (also see [116]). We also note that, a result of the above type will be obtained in Paragraph 3.2.2 (see Theorem 3.2.4) for a class of mixed problems with singular ϕ-Laplacian operator.

Example 2.3.15. If (2.2.3) holds, then there exists $\lambda^\star > 0$ such that, for every $\lambda > \lambda^\star$, the problem

$$\begin{cases} \mathcal{A}_k(x) = \lambda k \left(x^\theta(k) - x^{\theta+1}(k) \right), \quad (\forall)\ k \in \mathbb{Z}[1,T], \\ \\ x(0) - x(T+1) = 0 = \Delta x(0) - \Delta x(T) \end{cases}$$

has at least two positive solutions for any $\theta > p^+ - 1$.

In a similar way as above, for the Neumann problem (2.3.2) we have the following multiplicity result.

Theorem 2.3.16. *If there is some $\xi > 0$ such that $f(k,\cdot) > 0$ on $(0,\xi)$ and $f(k,\xi) = 0$, for all $k \in \mathbb{Z}[1,T]$ and (2.3.20) holds true, then there exists $\lambda^\star > 0$ such that for all $\lambda > \lambda^\star$, problem (2.3.2) has at least two positive solutions in X_N.*

2.3.3 Nontrivial solutions for problems (2.3.3) and (2.3.4)

Assuming a sign asymptotic behavior of the primitive $F(k,\cdot)$ near 0 and the Ambrosetti-Rabinowitz type condition (2.1.38), we prove in the sequel that problems (2.3.3) and (2.3.4) have at least two nontrivial solutions, for small enough values of the parameter λ.

In this view, we employ some ideas from [13], combined with specific technicalities due to the discrete and anisotropic character of the problems. The main ingredient will be Proposition 1.10. Throughout this paragraph, we assume that

$$\overline{q} < p^-. \tag{2.3.28}$$

Toward the application of Proposition 1.10, we first have to know that the energy functional $\Psi_{b,X}$, with $X = X_P$ (resp. $X = X_N$), satisfies the (PS) condition and that it is anticoercive on the subspace of constant functions.

Lemma 2.3.17. *Assume* (2.3.28). *If there are constants $\theta > p^+$ and $\rho > 0$ such that* (2.1.38) *holds true, then $\Psi_{b,X}$ satisfies the (PS) condition and*

$$\Psi_{b,X}(c) \to -\infty \quad \text{as } |c| \to \infty,\ c \in \mathbb{R},$$

for any $\lambda > 0$.

Proof. Let $\{x_n\} \subset X$ be a sequence for which $\{\Psi_{b,X}(x_n)\}$ is bounded and

$$\Psi'_{b,X}(x_n) \to 0, \qquad \text{as } n \to \infty. \tag{2.3.29}$$

Since X is finite dimensional, it suffices to prove that $\{x_n\}$ is bounded. Without loss of generality, we may assume that $\|x_n\|_{\underline{r},p(\cdot)} > 1$, for all $n \in \mathbb{N}$. Using (2.1.38), we deduce that, for all $n \in \mathbb{N}$, it holds (see (2.1.35))

$$\begin{aligned}\theta\mathcal{F}_X(x_n) - \langle \mathcal{F}'_X(x_n), x_n\rangle &= \sum_{k=1}^{T}[\theta F(k, x_n(k)) - x_n(k)f(k, x_n(k))]\\ &\le \sum_{k=1}^{T}\max_{|x|\le\rho}|\theta F(k,x) - xf(k,x)| =: C_1.\end{aligned} \tag{2.3.30}$$

From (2.3.28), (2.1.7), (2.3.5), (2.3.6) and (2.3.30), for all $n \in \mathbb{N}$, we have

$$\begin{aligned}&\left(\frac{\theta}{p^+} - 1\right)p^-\|x_n\|_{\underline{r},p(\cdot)}^{p^-}\\ &\quad\le \left(\frac{\theta}{p^+} - 1\right)\left[\sum_{k=1}^{T+1}|\Delta x_n(k-1)|^{p(k-1)} + \sum_{k=1}^{T} r(k)|x_n(k)|^{p(k)}\right]\\ &\quad\le \theta\left(\sum_{k=1}^{T+1}\frac{1}{p(k-1)}|\Delta x_n(k-1)|^{p(k-1)} + \sum_{k=1}^{T}\frac{r(k)}{p(k)}|x_n(k)|^{p(k)}\right)\\ &\qquad - \left(\sum_{k=1}^{T+1}|\Delta x_n(k-1)|^{p(k-1)} + \sum_{k=1}^{T} r(k)|x_n(k)|^{p(k)}\right)\\ &\quad= \theta\Psi_{b,X}(x_n) - \langle\Psi'_{b,X}(x_n), x_n\rangle + \sum_{k=1}^{T}[\theta F(k, x_n(k)) - x_n(k)f(k, x_n(k))]\\ &\qquad + \lambda\sum_{k=1}^{T}\left[\theta\frac{b(k)}{q(k)}|x_n(k)|^{q(k)} - b(k)|x_n(k)|^{q(k)}\right]\end{aligned}$$

$$\leq \theta\Psi_{b,X}(x_n) - \langle \Psi'_{b,X}(x_n), x_n\rangle + \lambda\left(\frac{\theta}{\underline{q}} - 1\right)\sum_{k=1}^{T} |b(k)||x_n(k)|^{q(k)} + C_1$$
$$\leq \theta\Psi_{b,X}(x_n) - \langle \Psi'_{b,X}(x_n), x_n\rangle + C_1$$
$$+ \lambda\|b\|_\infty\left(\frac{\theta}{\underline{q}} - 1\right)\sum_{k=0}^{T+1}\left(|x_n(k)|^{\overline{q}} + |x_n(k)|^{\underline{q}}\right).$$

Then, the equivalence of the norms on X yields

$$\left(\frac{\theta}{p^+} - 1\right)p^-\|x_n\|^{p^-}_{\underline{r},p(\cdot)} \leq \theta\Psi_{b,X}(x_n) - \langle \Psi'_{b,X}(x_n), x_n\rangle$$
$$+ \lambda C_2\|b\|_\infty\left(\frac{\theta}{\underline{q}} - 1\right)\left(\|x_n\|^{\overline{q}}_{\underline{r},p(\cdot)} + \|x_n\|^{\underline{q}}_{\underline{r},p(\cdot)}\right) + C_1$$
$$\leq \theta\Psi_{b,X}(x_n) + \|\Psi'_{b,X}(x_n)\|\|x_n\|_{\underline{r},p(\cdot)}$$
$$+ C_3\left(\|x_n\|^{\overline{q}}_{\underline{r},p(\cdot)} + \|x_n\|^{\underline{q}}_{\underline{r},p(\cdot)}\right) + C_1, \qquad (2.3.31)$$

with $C_2 > 0$ and $C_3 := \lambda C_2\|b\|_\infty\left(\frac{\theta}{\underline{q}} - 1\right) > 0$.

As $\{\Psi_{b,X}(x_n)\}$ is bounded and on account of (2.3.29), from (2.3.31), we get that $\{x_n\}$ is bounded and hence, $\Psi_{b,X}$ satisfies the (PS) condition.

Next, on account of Proposition 2.1.11, there exist $a_1, a_2 > 0$ such that

$$F(k,t) \geq a_1|t|^\theta - a_2, \quad (\forall)\ k \in \mathbb{Z}[1,T],\ (\forall)\ t \in \mathbb{R},$$

which yields

$$\Psi_{b,X}(c) = \sum_{k=1}^{T}\frac{r(k)}{p(k)}|c|^{p(k)} - \sum_{k=1}^{T}F(k,c) - \lambda\sum_{k=1}^{T}\frac{b(k)}{q(k)}|c|^{q(k)} \leq$$

$$\frac{\overline{r}T}{p^-}|c|^{p^+} - a_1T|c|^\theta + a_2T - \frac{\lambda|c|^{\underline{q}}}{\overline{q}}\sum_{\{k\in\mathbb{Z}[1,T];\ b(k)\geq 0\}} b(k) - \frac{\lambda|c|^{\overline{q}}}{\underline{q}}\sum_{\{k\in\mathbb{Z}[1,T];\ b(k)<0\}} b(k),$$

for all $c \in \mathbb{R}$, with $|c| > 1$. Then, since $\theta > p^+ > \overline{q} \geq \underline{q}$, the result follows. ■

Lemma 2.3.18. *Assume* (2.3.28) *and that* $\underline{b} > 0$. *If either*

$$\liminf_{t\to 0^-}\frac{F(k,t)}{|t|^{p(k)}} \geq 0, \quad (\forall)\ k \in \mathbb{Z}[1,T] \qquad (2.3.32)$$

or

$$\liminf_{t\to 0^+}\frac{F(k,t)}{t^{p(k)}} \geq 0, \quad (\forall)\ k \in \mathbb{Z}[1,T], \qquad (2.3.33)$$

then

$$\inf_{B_\zeta}\Psi_{b,X} < 0, \qquad (2.3.34)$$

for all $\zeta, \lambda > 0$.

Proof. First, note that $\underline{b} > 0$ means $b > 0$ on $\mathbb{Z}[1,T]$. Let us suppose that (2.3.32) holds true. A similar argument works under assumption (2.3.33). Condition (2.3.32) means that

$$\lim_{\varepsilon\to 0^+} \inf_{t\in(-\varepsilon,0)} \frac{F(k,t)}{|t|^{p(k)}} \geq 0, \quad (\forall)\ k \in \mathbb{Z}[1,T].$$

This yields the existence of some $\varepsilon_1 > 0$ so that

$$F(k,t) \geq -|t|^{p(k)}, \quad (\forall)\ k \in \mathbb{Z}[1,T],\ (\forall)\ t \in (-\varepsilon_1, 0], \tag{2.3.35}$$

and we may assume that $\zeta < \varepsilon_1$. For $c \in (-\zeta, 0) \subset (-\varepsilon_1, 0]$, using (2.3.35), (2.3.28) and $\underline{b} > 0$, we estimate $\Psi_{b,X}$ as follows

$$\begin{aligned}
\Psi_{b,X}(c) &= \sum_{k=1}^{T} \frac{r(k)}{p(k)}|c|^{p(k)} - \sum_{k=1}^{T} F(k,c) - \lambda \sum_{k=1}^{T} \frac{b(k)}{q(k)}|c|^{q(k)} \\
&\leq \frac{\overline{r}T}{p^-}|c|^{p^-} + T|c|^{p^-} - \frac{\lambda \underline{b}T}{\overline{q}}|c|^{\overline{q}} \\
&= |c|^{\overline{q}}T\left[\left(\frac{\overline{r}}{p^-} + 1\right)|c|^{p^- - \overline{q}} - \frac{\lambda \underline{b}}{\overline{q}}\right] < 0,
\end{aligned}$$

provided that $|c| \in (0,1)$ is small enough, which implies (2.3.34) and the proof is complete. ■

Lemma 2.3.19. *If*

$$\limsup_{|t|\to 0} \frac{p(k)F(k,t)}{|t|^{p(k)}} < \underline{r}, \quad (\forall)\ k \in \mathbb{Z}[1,T], \tag{2.3.36}$$

then there exist $\rho, \lambda_0 > 0$ *such that*

$$\inf_{\partial B_\rho} \Psi_{b,X} > 0, \tag{2.3.37}$$

for all $\lambda \in (0, \lambda_0)$.

Proof. As in the proof of Theorem 2.3.12, by the equivalence of the norms on X, for each $\eta > 0$, there is some $C_\eta > 0$ such that (2.3.25) holds true.

From (2.3.36), we can find constants $\sigma \in (0, \underline{r})$, $\rho \in (0,1)$ and $C_\sigma > 0$ such that

$$F(k,t) \leq \frac{\underline{r} - \sigma}{p(k)}|t|^{p(k)}, \quad (\forall)\ k \in \mathbb{Z}[1,T],\ (\forall)\ t \in \mathbb{R} \text{ with } |t| \leq \rho C_\sigma. \tag{2.3.38}$$

Let $x \in X$, with $\|x\|_{\sigma,p(\cdot)} = \rho$, be arbitrarily chosen. By (2.3.25) and (2.3.38), one has

$$F(k, x(k)) \leq \frac{\underline{r} - \sigma}{p(k)} |x(k)|^{p(k)}, \quad (\forall)\ k \in \mathbb{Z}[1, T],$$

which gives

$$\mathcal{F}_X(x) \leq (\underline{r} - \sigma) \sum_{k=1}^{T} \frac{1}{p(k)} |x(k)|^{p(k)}$$

and hence, from (2.1.8) and using again the equivalence of the norms on X, we get

$$\begin{aligned}
\Psi_{b,X}(x) &\geq \sum_{k=1}^{T+1} \frac{1}{p(k-1)} |\Delta x(k-1)|^{p(k-1)} + \underline{r} \sum_{k=1}^{T} \frac{1}{p(k)} |x(k)|^{p(k)} \\
&\quad + (\sigma - \underline{r}) \sum_{k=1}^{T} \frac{1}{p(k)} |x(k)|^{p(k)} - \lambda \sum_{k=1}^{T} \frac{b(k)}{q(k)} |x(k)|^{q(k)} \\
&\geq \|x\|_{\sigma,p(\cdot)}^{p^+} - \frac{\lambda \|b\|_\infty}{\underline{q}} \sum_{k=0}^{T+1} \left(|x(k)|^{\overline{q}} + |x(k)|^{\underline{q}} \right) \\
&\geq \|x\|_{\sigma,p(\cdot)}^{p^+} - \frac{\lambda C \|b\|_\infty}{\underline{q}} \left(\|x\|_{\sigma,p(\cdot)}^{\overline{q}} + \|x\|_{\sigma,p(\cdot)}^{\underline{q}} \right) \\
&= \rho^{p^+} - \frac{\lambda C \|b\|_\infty}{\underline{q}} \left(\rho^{\underline{q}} + \rho^{\overline{q}} \right),
\end{aligned}$$

with $C > 0$. Setting

$$\lambda_0 := \frac{\underline{q} \rho^{p^+}}{\|b\|_\infty \left(\rho^{\underline{q}} + \rho^{\overline{q}} \right) C} > 0,$$

one has

$$\Psi_{b,X}(x) \geq \frac{\|b\|_\infty \left(\rho^{\underline{q}} + \rho^{\overline{q}} \right) C}{\underline{q}} (\lambda_0 - \lambda) =: c_\lambda > 0,$$

for arbitrary $\lambda \in (0, \lambda_0)$ and (2.3.37) follows. ∎

Theorem 2.3.20. *Assume* (2.2.3), (2.3.28), (2.3.36) *and that* $\underline{b} > 0$. *If there are constants* $\theta > p^+$ *and* $\rho > 0$ *such that* (2.1.38) *holds and either* (2.3.32) *or* (2.3.33) *is satisfied, then there exists* $\lambda_0 > 0$ *such that problem* (2.3.3) *has at least two nontrivial solutions for any* $\lambda \in (0, \lambda_0)$.

Proof. The conclusion follows from Proposition 1.10, Lemmas 2.3.17 - 2.3.19 with $X = X_P$, and Proposition 2.3.1. ∎

In the same way, but with $X = X_N$ and Proposition 2.3.2 instead of Proposition 2.3.1, we have the following multiplicity result in the case of the Neumann problem (2.3.4).

Theorem 2.3.21. *Assume* (2.3.28), (2.3.36) *and that* $\underline{b} > 0$. *If there are constants* $\theta > p^+$ *and* $\rho > 0$ *such that* (2.1.38) *is satisfied and either* (2.3.32) *or* (2.3.33) *holds true, then there exists* $\lambda_0 > 0$ *such that problem* (2.3.4) *has at least two nontrivial solutions for any* $\lambda \in (0, \lambda_0)$.

Remark 2.3.22. (i) On account of Remark 1.11 (i), under the hypotheses of Theorem 2.3.20 (resp. Theorem 2.3.21), if, in addition, $f(k, \cdot)$ is odd for all $k \in \mathbb{Z}[1, T]$, then problem (2.3.3) (resp. (2.3.4)) has at least four nontrivial solutions for any $\lambda \in (0, \lambda_0)$.

(ii) If $q =$ constant, then it is not difficult to check that Theorem 2.3.20 and Theorem 2.3.21 remain valid with the weaker hypothesis $\sum_{k=1}^{T} b(k) > 0$ instead of $\underline{b} > 0$.

Example 2.3.23. If (2.3.28) holds true, $\theta > p^+$ and $\underline{b} > 0$, then there exists $\lambda_0 > 0$ such that the Neumann problem

$$\begin{cases} \mathcal{A}_k(x) = kh_\theta(x(k)) + \lambda b(k) h_{q(k)}(x(k)), & (\forall)\ k \in \mathbb{Z}[1, T], \\ \\ \Delta x(0) = 0 = \Delta x(T) \end{cases}$$

has at least four nontrivial solutions for any $\lambda \in (0, \lambda_0)$.

3

Existence and multiplicity results for singular ϕ-Laplacian

Contents

The first two sections of this chapter are mainly motivated by the existence and approximation of radial solutions for the nonlinear Dirichlet problem

$$\mathcal{M}v + g(|x|, v) = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R), \tag{3.1}$$

where $R > 0$, $\mathcal{B}(R) = \{x \in \mathbb{R}^N : |x| < R\}$, $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^N$, $\mathcal{M}$ is the *mean extrinsic curvature operator in Minkowski space*

$$\mathcal{M}v = \operatorname{div}\left(\frac{\nabla v}{\sqrt{1 - |\nabla v|^2}}\right)$$

and $g : [0, R] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Problems of type (3.1) are of interest in special relativity and related topics from Minkowski geometry; for such a discussion we refer, e.g., to [7], [26], [71] and the references therein.

Setting $r = |x|$ and $v(x) = u(r)$, problem (3.1) becomes

$$\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)' + r^{N-1}g(r,u) = 0, \qquad u'(0) = 0 = u(R) \tag{3.2}$$

and the solutions of (3.2) are just classical radial solutions of (3.1). Problems of type (3.2) were recently studied by C. Bereanu *et al.* in [19], [20] and I. Coelho *et al.* in [50]. More precisely, when $g(r,u) = \lambda u^{q-1} + \ell(r,u)$ with $q \in (1,2)$ and $\ell : [0,R]\times[0,\infty) \to [0,\infty)$ continuous, it is proved in [19] by a Leray-Schauder degree argument that (3.2) has at least one positive solution. Also, if $g(r,u) = \lambda u^{q-1}$, with $q \geq 2$, it is shown in the same paper [19] by a minimization procedure, that (3.2) has at least one positive solution for λ sufficiently large. In [20] it is proved, using the last mentioned result, in combination with the upper and lower solutions method and degree theory, that there exist $\Lambda > 0$ such that (3.2), with $g(r,u) = \lambda u^{q-1}$, $q > 2$, has zero, at least one and at least two positive solutions according to $\lambda \in (0,\Lambda)$, $\lambda = \Lambda$ and $\lambda > \Lambda$. Finally, if $g(r,u) = \lambda a(r)u^p + \mu b(r)u^q$, with $0 < p < 1 < q$ and functions $a, b : [0,R] \to \mathbb{R}$ are continuous and positive somewhere (in particular they may change sign), it is obtained in [50] by reduction to an equivalent non-singular one-dimensional problem that (3.2) has at least three positive solutions for all sufficiently large values of μ and all small positive values of λ. We also note that the results proved in [50] are similar to the corresponding ones obtained in [49] for problem (3.2) but with Dirichlet boundary conditions.

The study from Section 3.3 is motivated by the existence of positive solutions for the following Neumann problems with *attractive* restoring force

$$\mathcal{M}v + f(v) = h(|x|) \text{ in } \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\mathcal{A}, \tag{3.3}$$

or with *repulsive* restoring force

$$\mathcal{M}v - f(v) = h(|x|) \text{ in } \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\mathcal{A}, \tag{3.4}$$

where $0 \leq R_1 < R_2$, $\mathcal{A}$ is the annular domain $\{x \in \mathbb{R}^N : R_1 \leq |x| \leq R_2\}$ and $f : (0,+\infty) \to \mathbb{R}$ and $h : [R_1,R_2] \to \mathbb{R}$ are continuous functions. As usual, we have denoted by $\frac{\partial v}{\partial \nu}$ the outward normal derivative of v. Again, setting $r = |x|$ and $v(x) = u(r)$, the above problems (3.3) and (3.4) are reduced to the one-dimensional Neumann problems

$$\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)'+r^{N-1}f(u) = r^{N-1}h(r) \text{ in } [R_1, R_2], \quad u'(R_1) = 0 = u'(R_2) \tag{3.5}$$

respectively,

$$\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)'-r^{N-1}f(u) = r^{N-1}h(r) \text{ in } [R_1, R_2], \quad u'(R_1) = 0 = u'(R_2). \tag{3.6}$$

For other existence results concerning Neumann problems associated to prescribed mean curvature operator in Minkowski space $\mathcal{M}$ we refer the reader to [11]-[15], [103].

The rest of the chapter is organized as follows. In Section 3.1, we are concerned with numerical approximation of extremal (minimal/maximal) classical radial solutions for problem (3.1). The existence and multiplicity of solutions for problem (3.1) with some particular parameterized nonlinearity g are studied in Section 3.2. Section 3.3 is devoted to the Neumann problems with attractive and repulsive restoring forces.

3.1 Numerical extremal solutions for a problem with mixed boundary conditions

In this section we deal with the mixed boundary value problem

$$-(r^{N-1}\phi(u'))' = r^{N-1}g(r,u), \qquad u'(0) = 0 = u(R), \tag{3.1.1}$$

where $N \geq 1$ is an integer, $R > 0$, $g : [0, R] \times \mathbb{R} \to \mathbb{R}$ is a continuous function and $\phi : (-\eta, \eta) \to \mathbb{R}$ $(\eta < +\infty)$ is an increasing homeomorphism with $\phi(0) = 0$. Such an ϕ is called *singular*. Clearly, (3.2) is nothing else but a problem of type (3.1.1), with

$$\phi(y) = \frac{y}{\sqrt{1-y^2}} \quad (y \in (-1, 1)). \tag{3.1.2}$$

Boundary value problems involving singular ϕ-Laplacian operators have received a special attention mainly with respect to qualitative aspects, such as existence and multiplicity of the solutions (besides the above mentioned papers, we also refer to [8], [9], [22], [23], [36], [99], [102]). However, effective techniques to compute the solutions seems lagging behind. A first natural way to find numerical solutions is the classical finite differences method. But, due to the singular character of ϕ, this encounters serious damages as is pointed out in [82], where a simpler Dirichlet problem with singular ϕ-Laplacian is studied. That's why the authors develop a numerical approach relaying on a shooting technique combined with the classical Euler's method, which seems to work successfully for this type of singular problems. This will be the strategy we will also adopt in the sequel.

In the first paragraph, we apply the monotone iterative technique coupled with the method of lower and upper solutions in order to obtain two monotone sequences that uniformly converge to the extremal solutions of problem (3.1.1). Notice that the uniqueness of the solution will not be ensured by the assumptions on the data. Historically, the first steps in the theory of lower and upper solutions have been made in 1890 by E. Picard [114], where the existence of solutions for Partial Differential Equations is guaranteed from a monotone iterative technique. The reader is referred to [45] for a detailed study of monotone iterative techniques (also, see [38] and [39] for Neumann and periodic problems involving a ϕ-Laplacian operator).

If ϕ is an increasing diffeomorphism with $\phi(0) = 0$ and $\phi'(y) \geq d > 0$ for all $y \in (-\eta, \eta)$, we find in Paragraph 3.1.2 the approximate extremal solutions of problem (3.1.1). With this aim, we need to develop a numerical convergent algorithm for a problem of type

$$-(r^{N-1}\phi(u'))' = r^{N-1}\ell(r), \qquad u'(0) = 0 = u(R), \tag{3.1.3}$$

with $\ell : [0, R] \to \mathbb{R}$ continuous.

Throughout this section, C stands for the Banach space of all continuous functions defined on $[0, R]$ endowed with the usual sup-norm $\|\cdot\|_\infty$. Recall that by *a solution* of problem (3.1.1) we mean a function $u \in C^1 := C^1[0, R]$ with $\|u'\|_\infty < \eta$, such that $r^{N-1}\phi(u') \in C^1$ and (3.1.1) is satisfied.

Now, a fixed point operator is associated to problem (3.1.1) (see [10], [20], [47] for slightly different operators). In this view, setting

$$\sigma(r) := -\frac{1}{r^{N-1}} \quad (r > 0),$$

we define the linear operators

$$S : C \to C, \quad Su(r) = \sigma(r)\int_0^r t^{N-1}u(t)dt \quad (r \in (0, R]), \quad Su(0) = 0;$$

$$K : C \to C, \quad Ku(r) = \int_R^r u(t)dt, \quad (r \in [0, R]).$$

We have

$$\|Ku\|_\infty = \max_{r\in[0,R]} \left| \int_R^r u(t)dt \right| \le \max_{r\in[0,R]} \int_r^R |u(t)|dt \le R\|u\|_\infty$$

and hence K is bounded. Also, note that, since

$$|Su(r)| \le \frac{r}{N}\|u\|_\infty \to 0, \quad \text{as } r \to 0,$$

S is well defined. Moreover, S is compact. To see this, let $M \subset C$ be a bounded set. Then, clearly $S(M)$ is bounded, i.e.,

$$(\exists)\ C_1 > 0 \text{ such that } \|Su\|_\infty < C_1, \quad (\forall)\ u \in M.$$

Let $u \in M$. From Lagrange theorem, we infer

$$|Su(r_1) - Su(r_2)| \le |(Su)'||r_1 - r_2| \le \frac{2N-1}{N}|r_1 - r_2|\|u\|_\infty \le C_2|r_1 - r_2|,$$

where C_2 is a positive constant. Therefore, it is clear that $S(M)$ is equicontinuous. From the above and Arzelà-Ascoli's theorem, it follows that S is compact. In the same way, it is easy to see that K is also compact.

Note that, for any $\ell \in C$, the problem (3.1.3) has a unique solution u which is given by

$$u = K \circ \phi^{-1} \circ S \circ \ell.$$

Next, we denote by N_g the Nemytskii operator associated to g; recall that this is defined by

$$N_g(u) = g(\cdot, u(\cdot)), \quad (\forall)\ u \in C$$

and $N_g : C \to C$ is continuous and takes bounded sets into bounded sets.

Proposition 3.1.1. *(i) A function $u \in C$ is a solution of problem* (3.1.1) *if and only if it is a fixed point of the compact operator*

$$\mathcal{T} : C \to C, \qquad \mathcal{T} := K \circ \phi^{-1} \circ S \circ N_g.$$

(ii) Problem (3.1.1) *has at least a solution for any g continuous. If, in addition, $g(r, \cdot)$ is nonincreasing for each fixed $r \in [0, R]$, then the solution of* (3.1.1) *is unique.*

Proof. We use arguments from [10].

(i) This can be easily deduced from the properties of the operators composing $\mathcal{T}$.

(ii) Let $u \in C$ and $v = \mathcal{T}(u)$. Since $\phi^{-1} : \mathbb{R} \to (-\eta, \eta)$, one has

$$\|v'\|_\infty = \|\phi^{-1} \circ S \circ N_g(u)\|_\infty < \eta$$

and hence, we obtain

$$|v(r)| \leq \left| \int_R^r v'(t)dt \right| \leq \int_r^R |v'(t)|dt < \eta R. \tag{3.1.4}$$

Therefore, $\|v\|_\infty < \eta R$ and from Schauder fixed point theorem, we infer that there exist $u \in C$ such that $u = \mathcal{T}(u)$. Using (i), it follows that u is a solution of (3.1.1). For the second part of (ii), assume that u and w are solutions of problem (3.1.1) and that $u \neq w$. It follows from the boundary conditions that $E := \{r \in [0, R] : u'(r) \neq w'(r)\}$ has positive measure. Then, multiplying the identity

$$[r^{N-1}(\phi(u') - \phi(w'))]' = r^{N-1}[g(r, w) - g(r, u)]$$

by $u - w$, integrating over $[0, R]$, integrating by parts, using the boundary conditions, hypothesis (H_ϕ) and the fact that $g(r, \cdot)$ is nonincreasing for each fixed $r \in [0, R]$, we get

$$\begin{aligned} 0 &> -\int_E r^{N-1}[\phi(u'(r)) - \phi(w'(r))][u'(r) - w'(r)]dr \\ &= \int_0^R r^{N-1}[g(r, w(r)) - g(r, u(r))][u(r) - w(r)] \geq 0, \end{aligned}$$

a contradiction. ∎

Definition 3.1.2. Given a class of functions $\mathcal{U} \subset C$, a solution $u \in \mathcal{U}$ of problem (3.1.1) is called *minimal* (resp. *maximal*) in $\mathcal{U}$ if $u \leq v$ (resp. $v \leq u$), for any other solution $v \in \mathcal{U}$.

Definition 3.1.3. A *lower solution* of problem (3.1.1) is a function $\alpha \in C^1$ such that $\|\alpha'\|_\infty < \eta$, $r^{N-1}\phi(\alpha') \in C^1$ and

$$-(r^{N-1}\phi(\alpha'(r)))' \leq r^{N-1}g(r, \alpha(r)) \quad (r \in [0, R]), \quad \alpha(R) \leq 0. \tag{3.1.5}$$

Similarly, an *upper solution* of (3.1.1) is defined by reversing the inequalities in (3.1.5).

The rest of the section is organized as follows. Using the monotone iterative technique, we prove in Paragraph 3.1.1 the existence of minimal and maximal solutions in presence of well-ordered lower and upper solutions and we develop a numerical algorithm for theirs approximation in Paragraph

3.1.2. The algorithm combines the shooting method with Euler's method. Also, we provide numerical experiments confirming the theoretical aspects.

The results from this section are obtained in [83].

3.1.1 Extremal solutions with monotone iterative technique

In this paragraph we construct two monotone sequences which uniformly converge to the extremal solutions of (3.1.1) from an interval described by a lower and an upper solution, under the hypothesis

(H_ϕ) $\phi : (-\eta, \eta) \to \mathbb{R}$ $(0 < \eta < \infty)$ *is an increasing homeomorphism with* $\phi(0) = 0$.

We shall need the following comparison principle.

Proposition 3.1.4. *Let $M > 0$ and $v_1, v_2 \in C^1$ be such that $r^{N-1}\phi(v_i') \in C^1$ $(i = 1, 2)$. If*

$$-(r^{N-1}\phi(v_1'))' + Mr^{N-1}v_1 \leq -(r^{N-1}\phi(v_2'))' + Mr^{N-1}v_2 \quad \text{on } [0, R] \tag{3.1.6}$$

and $v_1(R) \leq v_2(R)$, then $v_1(r) \leq v_2(r)$ for all $r \in [0, R]$.

Proof. Suppose, by contradiction, that there exists some $r_0 \in [0, R)$ such that

$$\max_{[0,R]}(v_1 - v_2) = v_1(r_0) - v_2(r_0) > 0.$$

If $r_0 \in (0, R)$ then $v_1'(r_0) = v_2'(r_0)$ and there is a sequence $\{r_k\}$ in $(0, r_0)$ converging to r_0 such that $v_1'(r_k) - v_2'(r_k) \geq 0$. As ϕ is increasing, one has

$$r_k^{N-1}\phi(v_1'(r_k)) - r_0^{N-1}\phi(v_1'(r_0)) \geq r_k^{N-1}\phi(v_2'(r_k)) - r_0^{N-1}\phi(v_2'(r_0)),$$

implying that

$$(r^{N-1}\phi(v_1'(r)))'_{r=r_0} \leq (r^{N-1}\phi(v_2'(r)))'_{r=r_0}. \tag{3.1.7}$$

From (3.1.6) and (3.1.7), we obtain

$$0 < Mr_0^{N-1}(v_1(r_0) - v_2(r_0)) \leq (r^{N-1}\phi(v_1'(r)))'_{r=r_0} - (r^{N-1}\phi(v_2'(r)))'_{r=r_0} \leq 0,$$

a contradiction.

If $r_0 = 0$ then there exists $r_1 \in (0, R]$ such that $v_1(r) - v_2(r) > 0$ for all $r \in [0, r_1]$ and $v_1'(r_1) - v_2'(r_1) \leq 0$. By (H_ϕ) and (3.1.6), we get

$$0 \geq r_1^{N-1}\phi(v_1'(r_1)) - r_1^{N-1}\phi(v_2'(r_1)) \geq \int_0^{r_1} Mr^{N-1}(v_1(r) - v_2(r))dr > 0,$$

a contradiction, again. Consequently, $v_1(r) \leq v_2(r)$ for all $r \in [0, R]$. ∎

Now, we are in position to apply the monotone iterative method to provide the existence of extremal solutions for problem (3.1.1).

Theorem 3.1.5. *If* (3.1.1) *has a lower solution* α *and an upper solution* β *such that*

$$\alpha \leq \beta \quad \text{on } [0, R] \tag{3.1.8}$$

and for all $r \in [0, R]$, g *satisfies*

$$g(r, v) - g(r, w) \leq -L(v - w), \quad (\forall)\ v, w \in [\alpha(r), \beta(r)] \text{ with } v \leq w, \tag{3.1.9}$$

where L *is a positive constant, then problem* (3.1.1) *has a minimal solution* u_{min} *and a maximal solution* u_{max} *in* $\mathcal{U}_{\alpha,\beta} = \{v \in C : \alpha \leq v \leq \beta\}$.

Proof. We construct u_{min} by monotone iterations. Let $u_0 = \alpha$. On account of Proposition 3.1.1, we can define u_{n+1} $(n = 0, 1, 2, ...)$ as being the unique solution of the problem

$$\begin{cases} -(r^{N-1}\phi(u'_{n+1}))' + Lr^{N-1}u_{n+1} = r^{N-1}(Lu_n + g(r, u_n)), \\ \\ u'_{n+1}(0) = 0 = u_{n+1}(R). \end{cases} \tag{3.1.10}$$

Step 1. We claim that

$$\alpha(r) \leq u_n(r) \leq \beta(r) \quad (n = 0, 1, 2, ...), \quad (\forall)\ r \in [0, R]. \tag{3.1.11}$$

We proceed by induction. Clearly, (3.1.11) is valid for $n = 0$ by hypothesis (3.1.8). Then, assuming $\alpha \leq u_n \leq \beta$, we have to show that $\alpha \leq u_{n+1} \leq \beta$. From (3.1.10), (3.1.9) and because α is a lower solution of problem (3.1.1), we get

$$\begin{aligned} -(r^{N-1}\phi(u'_{n+1}(r)))' + Lr^{N-1}u_{n+1}(r) &= r^{N-1}(Lu_n(r) + g(r, u_n(r))) \\ &\geq r^{N-1}(L\alpha(r) + g(r, \alpha(r))) \\ &\geq -(r^{N-1}\phi(\alpha'(r)))' + Lr^{N-1}\alpha(r), \end{aligned}$$

and in view of Proposition 3.1.4, one has

$$\alpha(r) \leq u_{n+1}(r), \quad (\forall)\ r \in [0, R].$$

Similarly, for all $r \in [0, R]$, we deduce

$$-(r^{N-1}\phi(u'_{n+1}(r)))' + Lr^{N-1}u_{n+1}(r) \leq -(r^{N-1}\phi(\beta'(r)))' + Lr^{N-1}\beta(r)$$

and again from Proposition 3.1.4, we obtain

$$u_{n+1}(r) \leq \beta(r), \quad (\forall)\ r \in [0, R]$$

and (3.1.11) is proved, as claimed.

Step 2. We show that

$$u_n(r) \le u_{n+1}(r) \quad (n = 0, 1, 2, ...), \quad (\forall)\ r \in [0, R]. \tag{3.1.12}$$

Again, we proceed by induction. The case $n = 0$ follows from Step 1. Next, assume $u_{n-1} \le u_n$. From (3.1.9) - (3.1.11), we successively have

$$\begin{aligned} -(r^{N-1}\phi(u'_{n+1}(r)))' + Lr^{N-1}u_{n+1}(r) &= r^{N-1}(Lu_n(r) + g(r, u_n(r))) \\ &\ge r^{N-1}(Lu_{n-1}(r) + g(r, u_{n-1}(r))) \\ &= -(r^{N-1}\phi(u'_n(r)))' + Lr^{N-1}u_n(r), \end{aligned}$$

for all $r \in [0, R]$. Then (3.1.12) follows from Proposition 3.1.4.

Step 3 (Existence of u_{min}). On account of (3.1.11) and (3.1.12), we have

$$\alpha(r) = u_0(r) \le u_1(r) \le \dots u_n(r) \le u_{n+1}(r) \le \dots \le \beta(r), \quad (\forall)\ r \in [0, R].$$

Therefore

$$u_{min}(r) := \lim_{n\to\infty} u_n(r) \tag{3.1.13}$$

exists for all $r \in [0, R]$. Since $\{u_n\}_{n\ge1}$ and $\{u'_n\}_{n\ge1}$ are bounded in C, using Arzelà-Ascoli theorem we infer that $u_{min} \in C$. Then by (3.1.13) and Dini's theorem, $u_n \to u_{min}$, uniformly in C.

We rewrite problem (3.1.10) as

$$\begin{cases} -(r^{N-1}\phi(u'_{n+1}))' = r^{N-1}g_n(r, u_{n+1}), \\ \\ u'_{n+1}(0) = 0 = u_{n+1}(R), \end{cases} \tag{3.1.14}$$

with $g_n(r, s) := -Ls + Lu_n(r) + g(r, u_n(r))$. Associated to (3.1.14) is the operator $\mathcal{T}_n$, defined as in Proposition 3.1.1 (i), i.e.,

$$\mathcal{T}_n = K \circ \phi^{-1} \circ S \circ N_{g_n}.$$

On account of $\mathcal{T}_n(u_{n+1}) = u_{n+1}$, the continuity of the operator $K \circ \phi^{-1} \circ S$ and

$$N_{g_n}(u_{n+1}) = g_n(\cdot, u_{n+1}(\cdot)) \to g(\cdot, u_{min}(\cdot)) = N_g(u_{min})$$

yield $\mathcal{T}_n(u_{n+1}) \to \mathcal{T}(u_{min})$ in C. This implies that $\mathcal{T}(u_{min}) = u_{min}$ and so $u_{min} \in \mathcal{U}_{\alpha,\beta}$ is a solution of problem (3.1.1).

Step 4. If u is another solution of (3.1.1) in $\mathcal{U}_{\alpha,\beta}$, we shall show that

$$u_n(r) \le u(r) \quad (n = 0, 1, 2, ...), \quad (\forall)\ r \in [0, R]. \tag{3.1.15}$$

Clearly, (3.1.15) holds true for $n = 0$. By induction, if we assume that $u_n \leq u$, then we have to prove that $u_{n+1} \leq u$. As in Step 2, from (3.1.9) and (3.1.10), one has

$$\begin{aligned} -(r^{N-1}\phi(u'(r)))' + Lr^{N-1}u(r) &= r^{N-1}(Lu(r) + g(r, u(r))) \\ &\geq r^{N-1}(Lu_n(r) + g(r, u_n(r))) \\ &= -(r^{N-1}\phi(u'_{n+1}(r)))' + Lr^{N-1}u_{n+1}(r), \end{aligned}$$

and (3.1.15) follows from Proposition 3.1.4.

Hence, from (3.1.15), we get $u_{min} \leq u$, i.e., u_{min} is the minimal solution of problem (3.1.1) in $\mathcal{U}_{\alpha,\beta}$.

The proof of the existence of a decreasing sequence $\{v_n\}$, starting at $v_0 = \beta$, which converges uniformly to the maximal solution u_{max} of problem (3.1.1) in $\mathcal{U}_{\alpha,\beta}$, is similar. ■

Remark 3.1.6. It is worth to point out that if α is a lower solution and β an upper solution of problem (3.1.1) such that (3.1.8) holds true and for all $r \in [0, R]$, g satisfies

$$|g(r, v) - g(r, w)| \leq L|v - w|, \quad (\forall)\ v, w \in [\alpha(r), \beta(r)], \tag{3.1.16}$$

then (3.1.9) is fulfilled, hence problem (3.1.1) has minimal and maximal solutions in $\mathcal{U}_{\alpha,\beta}$. However, there are functions g verifying (3.1.9), which does not satisfy the L-Lipschitz condition (3.1.16) - see Example 3.1.9 from the next paragraph.

3.1.2 Numerical minimal and maximal solutions

Throughout this paragraph, instead of hypothesis (H_ϕ) we assume the stronger one, i.e.,

(H'_ϕ) $\phi : (-\eta, \eta) \to \mathbb{R}$ $(0 < \eta < \infty)$ *is an increasing diffeomorphism with* $\phi(0) = 0$ *and there is some* $d > 0$ *such that* $\phi'(y) \geq d$ *for all* $y \in (-\eta, \eta)$.

As we have seen in the proof of Theorem 3.1.5, finding the minimal and maximal solutions of problem (3.1.1) by the monotone iterative technique, requires to solve at step $n + 1$ a problem of the type (see (3.1.10)):

$$\begin{cases} -(r^{N-1}\phi(u'_{n+1}))' + Lr^{N-1}u_{n+1} = r^{N-1}\tilde{\ell}_n(r), \\ \\ u'_{n+1}(0) = 0 = u_{n+1}(R), \end{cases} \tag{3.1.17}$$

with $\tilde{\ell}_n(r) = Lu_n(r) + g(r, u_n(r))$ known from step n. The fixed point operator associated with (3.1.17) is (see Proposition 3.1.1 (i)):

$$\widetilde{\mathcal{T}}_{n+1} = K \circ \phi^{-1} \circ S \circ N_{\widetilde{g}_{n+1}},$$

where $\widetilde{g}_{n+1}(r,s) = -Ls + \tilde{\ell}_n(r)$ $((r,s) \in [0,R] \times \mathbb{R})$, with L entering in (3.1.9). If, in addition, one has that

$$L < \frac{2Nd}{R^2}, \tag{3.1.18}$$

then it is straightforward to see that $\widetilde{\mathcal{T}}_{n+1}$ is a contraction, hence by virtue of Banach's fixed point theorem, the sequence $\{u_{n+1}^k\}_{k\geq 0}$ defined by

$$u_{n+1}^{k+1} = \widetilde{\mathcal{T}}_{n+1}(u_{n+1}^k) \quad (k = 0, 1, 2, ...) \tag{3.1.19}$$

converges in C to the unique solution of problem (3.1.17), regardless of the initial value $u_{n+1}^0 \in C$. It is worth to point out that condition (3.1.18) does not imply the unique solvability of the original problem (3.1.1).

Now, setting $\bar{w} := u_{n+1}^{k+1}$, (3.1.19) is equivalent to

$$-(r^{N-1}\phi(\bar{w}'))' = r^{N-1}[-Lu_{n+1}^k(r) + \tilde{\ell}_n(r)], \quad \bar{w}'(0) = 0 = \bar{w}(R). \tag{3.1.20}$$

Thus, assuming (3.1.18), finding $\bar{w}$ in (3.1.20) reduces to solving a mixed problem of type:

$$-(r^{N-1}\phi(\bar{w}'))' = r^{N-1}f(r), \quad \bar{w}'(0) = 0 = \bar{w}(R), \tag{3.1.21}$$

with $f : [0,R] \to \mathbb{R}$ continuous.

In the sequel, combining Euler's method with a shooting technique, we derive an algorithm to find numerical solutions for problem (3.1.21). With this aim, setting $\psi := \phi^{-1}$ and $\bar{v} := -r^{N-1}\phi(\bar{w}')$, problem (3.1.21) is transformed into

$$\begin{cases} \bar{v}' = r^{N-1}f(r); \\ \bar{w}' = \psi\left(\sigma(r)\bar{v}(r)\right); \\ \bar{w}(R) = 0, \ \bar{v}(R) = a_*, \end{cases} \tag{3.1.22}$$

with

$$a_* = \int_0^R \zeta^{N-1} f(\zeta)\, d\zeta.$$

By the change of variable $t = R - r$ and setting $\bar{w}(r) = w(t)$, $\bar{v}(r) = v(t)$ in (3.1.22), we arrive to the initial value problem

$$\begin{cases} v' = -\gamma(t); \\ w' = -\psi\left(\sigma(R-t)v(t)\right); \\ w(0) = 0, \ v(0) = a_*, \end{cases} \tag{3.1.23}$$

where, we have denoted

$$\gamma(t) := (R-t)^{N-1} f(R-t). \tag{3.1.24}$$

Obviously, a_* can be expressed as

$$a_* = \int_0^R \gamma(t)\ dt. \tag{3.1.25}$$

Notice that both of the systems (3.1.22) and (3.1.23) have unique solutions.

Next, let $m \in \mathbb{N}$ ($m \geq 2$), $h = R/m$ and $t_i = ih$, $r_i = (m-i)h$ for $i \in \{0, 1, \ldots, m\}$. Applying the standard Euler's method (see e.g. [37]) to problem (3.1.23) and the rectangles quadrature formula to approximate a_* in (3.1.25), we get

$$\begin{cases} v_{i+1} = v_i - h\gamma(t_i), \ v_0 = a_m; \\ \\ w_{i+1} = w_i - h\psi\left(\sigma(R-t_i)v_i\right), \ w_0 = 0, \end{cases} \tag{3.1.26}$$

for $i \in \{0, 1, \ldots, m-1\}$, with

$$a_m = h \sum_{j=0}^{m-1} \gamma(t_j). \tag{3.1.27}$$

From (3.1.26), it follows

$$v_i = a_m - h \sum_{j=0}^{i-1} \gamma(t_j), \quad (i \in \{1, 2, \ldots, m\}),$$

which implies that

$$w_1 = -h\psi(\sigma(R)a_m) \tag{3.1.28}$$

and

$$\begin{aligned} w_i &= w_{i-1} - h\psi\left(\sigma(R-t_{i-1})v_{i-1}\right) \\ &= w_{i-1} - h\psi\left(\sigma(R-t_{i-1})\left(a_m - h\sum_{j=0}^{i-2}\gamma(t_j)\right)\right), \end{aligned} \tag{3.1.29}$$

for $i \in \{2, 3, \ldots, m\}$.

Now, to compute the approximate values $\bar{w}_i$ of $\bar{w}(r_i)$ ($\bar{w}$ = the exact solution of problem (3.1.21)) we have the following:

Algorithm A.

Step 1: use (3.1.27) to find a_m;

Step 2: compute w_1 with (3.1.28) and w_i by means of (3.1.29) for $i \in \{2, 3, \ldots, m\}$;

Step 3: $\bar{w}_i := w_{m-i}$ for $i \in \{0, 1, \ldots, m\}$.

The convergence of this algorithm is proved in the following theorem.

Theorem 3.1.7. *Assume* (H'_ϕ) *and* $f \in C$. *Then it holds*

$$\lim_{m\to\infty}\left(\max_{0\le i\le m}|\bar{w}(r_i) - \bar{w}_i|\right) = 0. \tag{3.1.30}$$

Proof. First, it is clear that (3.1.30) means nothing else but

$$\lim_{m\to\infty}\left(\max_{0\le i\le m}|w(t_i) - w_i|\right) = 0. \tag{3.1.31}$$

Also, since $f \in C$, there is a constant $M > 0$ such that

$$|f(s)| \le M, \quad (\forall)\ s \in [0, R]. \tag{3.1.32}$$

From (3.1.23), one has

$$v(t) = a_* - \int_0^t \gamma(\zeta)\,d\zeta$$

and

$$w(t) = -\int_0^t \psi\left(\sigma(R-\tau)v(\tau)\right)\,d\tau.$$

By the mean value theorem for Riemann integral, we obtain

$$\begin{aligned} w(t_i) - w(t_{i-1}) &= -\int_{t_{i-1}}^{t_i} \psi\left(\sigma(R-\tau)v(\tau)\right)\,d\tau \\ &= -h\psi\left(\sigma(R-\tau_i)v(\tau_i)\right) \\ &= -h\psi\left(\sigma(R-\tau_i)\left(a_* - \int_0^{\tau_i}\gamma(\zeta)\,d\zeta\right)\right), \end{aligned} \tag{3.1.33}$$

with some $\tau_i \in (t_{i-1}, t_i)$. Therefore, with $\varphi : [0, R] \to \mathbb{R}$ given by

$$\varphi(t) = \sigma(R-t)\int_t^R \gamma(s)\,ds \quad (t \in [0, R)), \quad \varphi(R) = 0,$$

from (3.1.33), (3.1.25), (3.1.29) and (3.1.27), we successively have

$$w(t_i) - w(t_{i-1}) = -h\psi\left(\varphi(\tau_i)\right) = (w_i - w_{i-1}) \\ - h\left[\psi\left(\varphi(\tau_i)\right) - \psi\left(\sigma(R - t_{i-1})h\sum_{j=i-1}^{m-1}\gamma(t_j)\right)\right],$$

for all $i \in \{1, 2, \ldots, m\}$. Then, using that

$$|\psi'(\xi)| \leq \frac{1}{d}, \quad \text{for all } \xi \in \mathbb{R} \quad \big(\text{see hypothesis } (H'_\phi)\big),$$

we get the estimate

$$|w(t_i) - w_i| \leq |w(t_{i-1}) - w_{i-1}| + \frac{h}{d}\left|\varphi(\tau_i) - \sigma(R - t_{i-1})h\sum_{j=i-1}^{m-1}\gamma(t_j)\right|, \tag{3.1.34}$$

for $i \in \{1, 2, \ldots, m\}$.

Now, let $\varepsilon > 0$ be arbitrary. Since φ and f are uniformly continuous, we can find $\delta_\varepsilon > 0$ such that

$$|\varphi(s') - \varphi(s'')| < d\varepsilon \tag{3.1.35}$$

and

$$|f(R - s') - f(R - s'')| < d\varepsilon, \tag{3.1.36}$$

for all $s', s'' \in [0, R]$ with $|s' - s''| < \delta_\varepsilon$.

If $N \geq 2$, we choose $m_\varepsilon \in \mathbb{N}$, so that

$$h = \frac{R}{m} < \min\left\{\delta_\varepsilon, \frac{d\varepsilon}{M(N-1)}\right\} \tag{3.1.37}$$

for all $m \geq m_\varepsilon$. Denoting

$$q_i := |w(t_i) - w_i| \quad (i \in \{1, 2, \ldots, m\}),$$

from (3.1.34) and (3.1.35), it follows

$$q_i \leq q_{i-1} + \frac{h}{d}\left(|\varphi(\tau_i) - \varphi(t_{i-1})| + \left|\varphi(t_{i-1}) - \sigma(R - t_{i-1})h\sum_{j=i-1}^{m-1}\gamma(t_j)\right|\right) \\ < q_{i-1} + h\varepsilon + \frac{h}{d(R - t_{i-1})^{N-1}}\left|\int_{t_{i-1}}^{R}\gamma(\zeta)d\zeta - h\sum_{j=i-1}^{m-1}\gamma(t_j)\right|.$$

Using again the mean value theorem for Riemann integral, we get

$$q_i < q_{i-1} + h\varepsilon + \frac{h^2}{d(R-t_{i-1})^{N-1}} \sum_{j=i-1}^{m-1} |\gamma(\rho_j) - \gamma(t_j)|, \qquad (3.1.38)$$

where $\rho_j \in (t_j, t_{j+1})$. On the other hand, for any $j \in \{i-1, \ldots, m-1\}$, one has

$$|(R-\rho_j)^{N-1} - (R-t_j)^{N-1}| \leq (N-1)(R-\varsigma_j)^{N-2}(\rho_j - t_j), \qquad (3.1.39)$$

with some $\varsigma_j \in (t_j, \rho_j)$. Then, taking into account that

$$R - t_{i-1} = h(m-i+1), \quad (i \in \{1, 2, \ldots, m\}),$$

from (3.1.36), (3.1.32) and (3.1.39), we obtain

$$\begin{aligned} |\gamma(\rho_j) - \gamma(t_j)| &= \left|(R-\rho_j)^{N-1} f(R-\rho_j) - (R-t_j)^{N-1} f(R-t_j)\right| \\ &\leq (R-\rho_j)^{N-1} |f(R-\rho_j) - f(R-t_j)| \\ &\quad + |f(R-t_j)||(R-\rho_j)^{N-1} - (R-t_j)^{N-1}| \\ &\leq (R-t_j)^{N-1} d\varepsilon + M(N-1)(R-\varsigma_j)^{N-2}(\rho_j - t_j) \\ &\leq (R-t_{i-1})^{N-1} d\varepsilon + M(N-1)(R-t_{i-1})^{N-2}(t_{j+1} - t_j) \\ &= (R-t_{i-1})^{N-1}\left(d\varepsilon + \frac{M(N-1)}{m-i+1}\right), \end{aligned}$$

which yields

$$q_i < q_{i-1} + h\varepsilon + h^2(m-i+1)\varepsilon + \frac{h^2}{d} M(N-1) < q_{i-1} + h\varepsilon + hR\varepsilon + h\varepsilon.$$

Hence, for all $m \geq m_\varepsilon$ and $i \in \{1, 2, \ldots, m\}$, since $q_0 = 0$ and $ih \leq R$, it follows

$$q_i < q_{i-1} + h(R+2)\varepsilon < \cdots < ih(R+2)\varepsilon \leq R(R+2)\varepsilon$$

and obviously (3.1.31) holds true.

If $N = 1$, we can find some $m_\varepsilon \in \mathbb{N}$, such that

$$h = R/m < \delta_\varepsilon \qquad (3.1.40)$$

for all $m \geq m_\varepsilon$ and, (3.1.38) becomes

$$q_i < q_{i-1} + h\varepsilon + \frac{h^2}{d} \sum_{j=i-1}^{m-1} |f(R-\rho_j) - f(R-t_j)|$$

and from (3.1.36), one gets

$$q_i < q_{i-1} + h\varepsilon + h^2(m - i + 1)\varepsilon \le q_{i-1} + h\varepsilon + hR\varepsilon.$$

Thus, as $q_0 = 0$ and $ih \le R$, we obtain

$$q_i < q_{i-1} + h(R+1)\varepsilon < \cdots < ih(R+1)\varepsilon \le R(R+1)\varepsilon,$$

for all $m \ge m_\varepsilon$ and $i \in \{1, 2, \ldots, m\}$. This clearly implies (3.1.31) and the proof is complete. ■

Example 3.1.8. Let $N \ge 1$, ϕ be given in (3.1.2) and $f : [0,1] \to \mathbb{R}$ be defined by

$$f(r) = \frac{N}{\sqrt{4 - r^2}} + \frac{r^2}{\sqrt{(4 - r^2)^3}}, \quad (\forall)\ r \in [0,1].$$

We know that

$$u(r) = \frac{1}{4}(1 - r^2)$$

is the exact solution of problem

$$-\left(r^{N-1}\frac{u'}{\sqrt{1 - u'^2}}\right)' = r^{N-1}f(r), \qquad u'(0) = 0 = u(1).$$

For $N = 1, 2, 6, 13$ and different values of the discretization parameter m, we give in Table 3.1.1 the values of the errors computed by

$$\max_{0 \le i \le m} |\bar{w}(r_i) - \bar{w}_i| \quad \text{(see (3.1.30))}.$$

As it can be seen from the table, for a fixed m the error grows as N does. This is in connection with the convergence assumptions on the parameter h (see (3.1.37)).

N	$m = 16$	$m = 32$	$m = 64$	$m = 128$
1	0.0177	0.0088	0.0044	0.0022
2	0.0452	0.0227	0.0114	0.0057
6	0.1129	0.0549	0.0268	0.0132
13	0.2356	0.1102	0.0512	0.0247

Table 3.1.1: Approximation errors for the test problem.

Using Algorithm A, in Figure 3.1.1 we present the graphics of the exact solution and its corresponding approximations. The numerical experiments are performed for $N = 2, 6$ and $m = 16$, respectively $m = 32$.

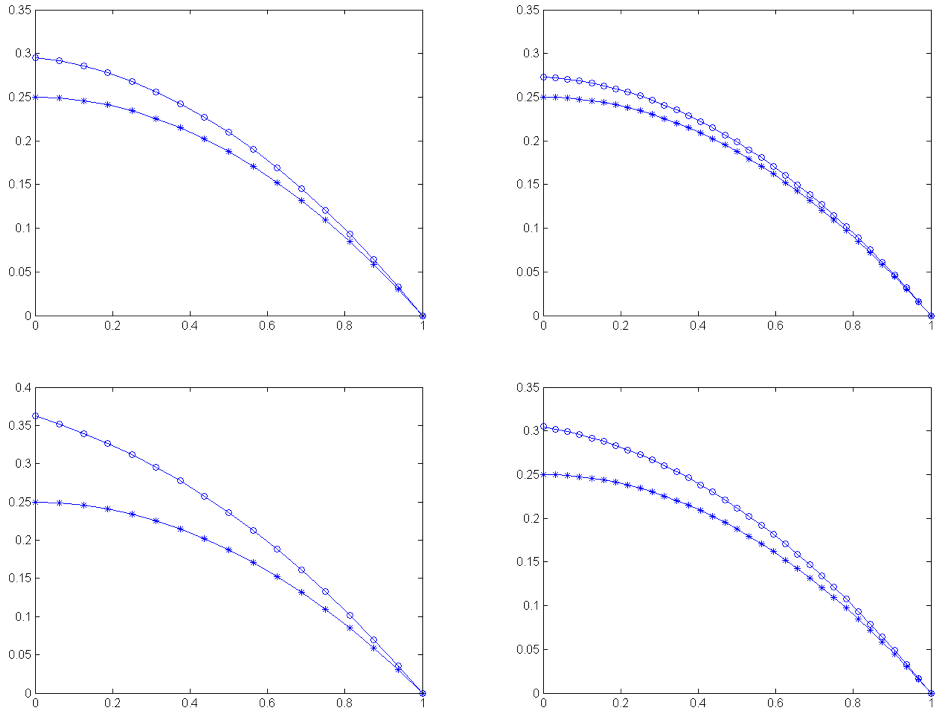

Figure 3.1.1: Exact $(- * -)$ and approximate $(- \circ -)$ solutions for the test problem (left to right) $m = 16$, $m = 32$; Top: $N = 2$; Bottom: $N = 6$.

At this stage we give the following algorithm to find the approximate minimal (resp. maximal) solution of problem (3.1.1).

Algorithm B.

- Given g continuous, ϕ satisfying (H'_ϕ), one assume that the hypotheses of Theorem 3.1.5 and (3.1.18) are fulfilled.
- Let $m \in \mathbb{N}$ $(m \geq 2)$ be fixed; we take $u_0 := \alpha$ (resp. $u_0 := \beta$).
- The function u_{n+1} $(n = 0, 1, \ldots)$ is approximated as follows:
  - Let $u^0_{n+1} \in C$ be arbitrary.
  - We compute, at step $k + 1$ $(k = 0, 1, \ldots)$, the approximate values of $u^{k+1}_{n+1}(r_i)$ $(i = \{0, 1, \ldots, m\})$ using Algorithm A, with f entering in (3.1.24) given by $f(r) = -Lu^k_{n+1}(r) + Lu_n(r) + g(r, u_n(r))$.

Example 3.1.9. To illustrate the above theoretical aspects, we consider the problem

$$-\left(r^{N-1}\frac{u'}{\sqrt{1-u'^2}}\right)' = r^{N-1}g(r,u), \qquad u'(0) = 0 = u(1), \qquad (3.1.41)$$

where $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ is defined by

$$g(r,u) = \begin{cases} sign(u-1)\sqrt{|u-1|}, & u \in (0,2); \\ -u+3, & u \geq 2; \\ -1-u/2, & u \leq 0, \end{cases} \qquad (\forall)\ r \in [0,1].$$

It is easy to check that $\alpha(r) = -2$ and $\beta(r) = 3$ ($r \in [0,1]$) are a lower, respectively an upper solution of problem (3.1.41) and that g fulfills condition (3.1.9) with $L = 1$. Clearly (3.1.18) is satisfied; ϕ is given in (3.1.2) and $d = 1$ in hypothesis (H'_ϕ). Also, note that g does not verify condition (3.1.16), hence in this case it can not be applied the classical method of successive approximations directly on the problem. We apply Algorithm B with the following stopping criteria

- for the *inner* iteration

$$|u_{n+1}^{k+1} - u_{n+1}^{k}| \leq 10^{-15}$$

- for the *outer* iteration

$$|u_{n+1} - u_n| \leq 10^{-10}$$

To find the approximate minimal (resp. maximal) solution, for the outer iteration we consider $u_0 = -2$ (resp. $u_0 = 3$), while for the inner iterations we consider $u_{n+1}^0 = 0$ ($n = 0, 1, \ldots$). In the case of maximal solution, in Table 3.1.2 we give the number of outer iterations and the average number of inner iterations (in brackets), for different values of N and m (similar values were obtained in the case of minimal solution).

N	$m=8$	$m=16$	$m=32$	$m=64$
1	12(22)	12(21)	12(20)	11(20)
2	12(21)	11(20)	11(19)	11(19)
6	10(17)	10(15)	9(14)	9(14)
11	9(15)	8(13)	8(12)	8(12)

Table 3.1.2: The number of outer (average inner) iterations for the test problem (3.1.41).

Our numerical experiments showed that problem (3.1.41) seems to have a unique solution between α and β, which is both minimal and maximal. The behavior of Algorithm B is different if we look for a minimal, respectively a maximal solution, but the corresponding sequences $(u_{n+1})_{n\geq 0}$ converges to the same function; in Figure 3.1.2 we plotted these sequences for $N = 2$ and $m = 32$. Also, in Figure 3.1.3 we give the graphics of the approximate (unique) solution of (3.1.41), which is computed for $m = 32$, $N = 2, 6$.

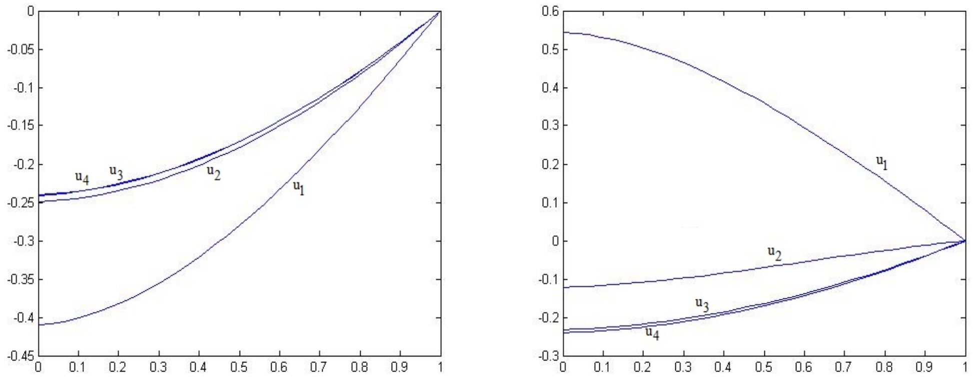

Figure 3.1.2: Example 3.3, case $N = 2, m = 32$; Left: sequence $(u_{n+1})_{n\geq 0}$ for $u_0 = -2$; Right: sequence $(u_{n+1})_{n\geq 0}$ for $u_0 = 3$.

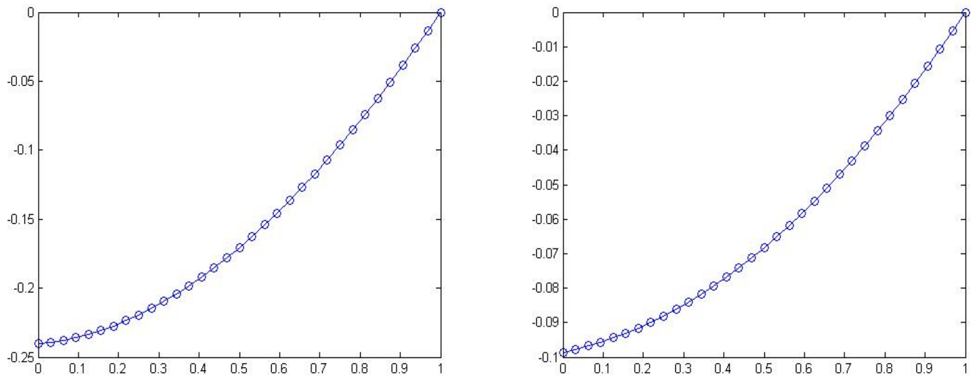

Figure 3.1.3: The approximate (unique) solution of the test problem (3.1.41); case $m = 32$; (left to right) $N = 2$ and $N = 6$.

3.2 Nontrivial solutions for a class of one-parameter mixed problems

In this section we deal with the mixed boundary value problem

$$[r^{N-1}\phi(u')]' = r^{N-1}[\mu(r)u^{q-1} - \lambda b(r,u)] \text{ in } [0,R], \quad u'(0) = 0 = u(R), \tag{3.2.1}$$

where $N \geq 1$, $R > 0$ and λ is a positive parameter. We assume the following hypotheses on the data:

(H_Φ) *$\phi : (-\eta, \eta) \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$, so that there exists $\Phi : [-\eta, \eta] \to \mathbb{R}$ continuous, of class C^1 on $(-\eta, \eta)$, with $\Phi(0) = 0$ and satisfying $\Phi' = \phi$;*

(H_b) *the continuous function $b : [0, R] \times [0, A] \to \mathbb{R}$ is such that $b(r, 0) = 0 = b(r, A)$ for all $r \in [0, R]$ and $b(r, s) > 0$ for all $(r, s) \in (0, R) \times (0, A)$;*

$(H_{\mu,q})$ *$\mu : [0, R] \to \mathbb{R}$ is continuous, positive on $(0, R)$ and $q > 1$ is a fixed real number.*

Hypothesis (H_Φ) was introduced in [23] (see also [13], [14], [19], [36]) and it clearly implies that Φ is strictly convex and $\Phi(s) \geq 0$ for all $s \in [-\eta, \eta]$. Notice that (3.2) is nothing else but a problem of type (3.2.1) with ϕ from (3.1.2), $\Phi(s) = 1 - \sqrt{1 - s^2}$ and the λ-parameterized nonlinearity $g(r, u) = \lambda b(r, u) - \mu(r)u^{q-1}$.

Using a variational approach, for large enough values of the parameter λ we provide sufficient conditions ensuring the existence of at least one or at least two nontrivial solutions for problem (3.2.1). The arguments which we employ to prove the main result of the section are similar to those from the case of Neumann and periodic problems involving the discrete $p(\cdot)$-Laplacian from Paragraph 2.3.2.

The section is organized as follows. In the first paragraph we introduce the variational setting for problem (3.2.1). The main result is proved in Paragraph 3.2.2; some examples of applications are also provided.

The results in this section are obtained in [17].

3.2.1 A variational approach

First, we introduce the function $f : [0, R] \times \mathbb{R} \to [0, \infty)$ defined by

$$f(r, s) = \begin{cases} 0, & \text{if } s < 0 \text{ or } s > A, \\ b(r, s), & \text{if } s \in [0, A] \end{cases} \tag{3.2.2}$$

and consider the auxiliary problem

$$[r^{N-1}\phi(u')]' = r^{N-1}[\mu(r)|u|^{q-2}u - \lambda f(r, u)] \ \text{ in } [0, R], \quad u'(0) = 0 = u(R). \tag{3.2.3}$$

Proposition 3.2.1. *If u is a solution of problem* (3.2.3) *then $0 \leq u \leq A$, hence u solves problem* (3.2.1).

Proof. Suppose that there exists some $r_0 \in [0, R)$ such that

$$\min_{[0,R]} u = u(r_0) < 0.$$

If $r_0 \in (0, R)$ then $u'(r_0) = 0$ and there is a sequence $\{r_k\}$ in $(0, r_0)$ converging to r_0 such that $u'(r_k) \leq 0$. Then

$$\frac{r_k^{N-1}\phi(u'(r_k)) - r_0^{N-1}\phi(u'(r_0))}{r_k - r_0} = \frac{r_k^{N-1}\phi(u'(r_k))}{r_k - r_0} \geq 0,$$

implying that

$$[r^{N-1}\phi(u'(r))]'_{r=r_0} \geq 0.$$

This yields the contradiction

$$0 \leq [r^{N-1}\phi(u'(r))]'_{r=r_0} = r_0^{N-1}\mu(r_0)|u(r_0)|^{q-2}u(r_0) < 0.$$

If $r_0 = 0$ then there exists $r_1 \in (0, R)$ such that $u(r) < 0$ for all $r \in [0, r_1]$ and $u'(r_1) \geq 0$. Integrating the equation in (3.2.3) from 0 to r_1, we obtain

$$0 \leq r_1^{N-1}\phi(u'(r_1)) = \int_0^{r_1} r^{N-1}\mu(r)|u(r)|^{q-2}u(r)dr < 0$$

a contradiction, again. Consequently, $u(r) \geq 0$ for all $r \in [0, R]$.

A quite similar reasoning shows that $u(r) \leq A$ for all $r \in [0, R]$ and the proof is complete. ■

Remark 3.2.2. (i) According to [20], the functions $\alpha = 0$ and $\beta = A$ are lower, respectively upper solutions for problem (3.2.3).

(ii) The reader will emphasize that if instead of (H_b) it is assumed "$b : [0, R] \times [0, \infty) \to [0, \infty)$ *is such that* $b(r, 0) = 0$ *for all* $r \in [0, R]$" then, changing f from (3.2.2) into

$$f(r,s) = \begin{cases} 0, & \text{if } s < 0, \\ b(r,s), & \text{if } s \geq 0, \end{cases} \qquad ((r,s) \in [0,R] \times \mathbb{R}), \tag{3.2.4}$$

each solution u of problem (3.2.3) is ≥ 0, hence u solves problem (3.2.1).

Next, using the general setting from Section 3 in [19], a variational approach is introduced for problem (3.2.3). With this aim, as in the previous section the space $C := C[0, R]$ will be endowed with the usual supremum norm $\|\cdot\|_\infty$ and the corresponding open ball of center 0 and radius $\sigma > 0$ will be denoted by B_σ. We set $W^{1,\infty} := W^{1,\infty}(0, R)$.

According to [15], the convex set

$$K_0 := \{v \in W^{1,\infty} \ : \ \|v'\|_\infty \leq \eta, \ v(R) = 0\}$$

is closed in C. On the other hand, as (see (3.1.4))

$$\|v\|_\infty \le \eta R \quad \text{for all } \ v \in K_0, \tag{3.2.5}$$

K_0 is bounded in $W^{1,\infty}$. Then, by the compactness of the embedding $W^{1,\infty} \subset C$, we infer that K_0 is compact in C.

Let $\Psi : C \to (-\infty, +\infty]$ be given by

$$\Psi(v) = \begin{cases} \displaystyle\int_0^R r^{N-1}\Phi(v')\,dr, & \text{if } v \in K_0, \\ +\infty, & \text{if } v \in C \setminus K_0. \end{cases}$$

Obviously, Ψ is proper and convex. Also, as shown in [15], Ψ is l.s.c on C.

Setting

$$F(r,s) = \int_0^s f(r,\xi)d\xi, \quad (r,s) \in [0,R] \times \mathbb{R},$$

we define $\mathcal{F}_\lambda : C \to \mathbb{R}$ by

$$\mathcal{F}_\lambda(v) = \int_0^R r^{N-1}\left[\frac{\mu(r)}{q}|v|^q - \lambda F(r,v)\right]dr, \quad v \in C,$$

which is of class C^1 on C. Then, the energy functional $I_\lambda := \Psi + \mathcal{F}_\lambda$ has the structure required by Szulkin's critical point theory. According to Paragraph 3 in Preliminaries, a function $u \in C$ is a critical point of I_λ if $u \in K_0$ and

$$\Psi(v) - \Psi(u) + \langle \mathcal{F}_\lambda'(u), v - u\rangle \ge 0, \quad \text{for all } \ v \in C.$$

Also, I_λ is said to satisfy the (PS) condition if any sequence $\{u_n\} \subset C$ such that $I_\lambda(u_n) \to c \in \mathbb{R}$ and

$$\Psi(v) - \Psi(u_n) + \langle \mathcal{F}_\lambda'(u_n), v - u_n\rangle \ge -\varepsilon_n\|v - u_n\|_\infty, \quad \text{for all } \ v \in C,$$

where $\varepsilon_n \to 0$, possesses a convergent subsequence.

Lemma 3.2.3. *The functional I_λ satisfies the (PS) condition and each critical point of I_λ is a solution of problem* (3.2.1).

Proof. The compactness of $K_0 \subset C$ implies that I_λ satisfies the (PS) condition, while from [19, Proposition 4] and Proposition 3.2.1 we have that each critical point of I_λ is a solution of problem (3.2.1). ■

3.2.2 Nontrivial solutions for problem (3.2.1)

The main result in this section is stated in the following theorem. The tools used in the proof will be Theorems 1.14 and 1.15.

Theorem 3.2.4. *(i) Under the assumptions* (H_Φ), (H_b) *and* $(H_{\mu,q})$, *there exists* $\Lambda > 0$ *such that problem* (3.2.1) *has at least one nontrivial solution for all* $\lambda > \Lambda$.

(ii) If, in addition, one has $\mu_M := \min_{[0,R]} \mu > 0$ *and* b *satisfies*

$$\lim_{s \to 0+} \frac{b(r,s)}{s^{q-1}} = 0 \quad \textit{uniformly with } r \in [0,R], \tag{3.2.6}$$

then, problem (3.2.1) *has at least two nontrivial solutions for all* $\lambda > \Lambda$.

Proof. (i) Let $u_0 \in K_0$ be defined by

$$u_0(r) = \min\left\{A, \frac{2R\eta}{\pi}\right\} \cos\frac{\pi r}{2R} \quad (r \in [0,R]).$$

It is clear that

$$F(r, u_0(r)) = \int_0^{u_0(r)} b(r,\xi)d\xi > 0 \quad \text{for all } r \in (0,R).$$

We take

$$\Lambda := \frac{\Psi(u_0) + \frac{1}{q}\int_0^R r^{N-1}\mu(r)u_0^q}{\int_0^R r^{N-1}F(r,u_0)}.$$

Then, for $\lambda > \Lambda$, from

$$I_\lambda(u_0) = \Psi(u_0) + \frac{1}{q}\int_0^R r^{N-1}\mu(r)u_0^q dr - \lambda\int_0^R r^{N-1}F(r,u_0)dr < 0$$

it follows that

$$c_\lambda := \inf_C I_\lambda < 0.$$

Since I_λ is bounded from below (this follows from (3.2.5)) and satisfies the (PS) condition (Lemma 3.2.3), by virtue of Theorem 1.15, c_λ is a critical value of I_λ. Consequently, there exists a critical point $u_{\lambda,1}$ of I_λ such that $I_\lambda(u_{\lambda,1}) = c_\lambda < 0$, for all $\lambda > \Lambda$. Clearly, $u_{\lambda,1} \neq 0$ and from Lemma 3.2.3 we have that $u_{\lambda,1}$ is a solution of problem (3.2.1).

(ii) Let $\lambda > \Lambda$ be fixed. We shall produce a second nontrivial critical point of I_λ by the Mountain Pass Theorem. To do this, it suffices to prove that

$$\inf_{K_0 \cap \partial B_\rho} I_\lambda > 0, \tag{3.2.7}$$

for some $\rho \in (0, \|u_{\lambda,1}\|_\infty/2)$.

Using (3.2.6) we can find $\delta > 0$, such that

$$F(r,s) \le \frac{\mu_M}{2q\lambda}|s|^q \quad \text{for all } (r,s) \in [0,R] \times [0,\delta].$$

Then, for $u \in C$ with $\|u\|_\infty \le \delta$, we have

$$\begin{aligned} I_\lambda(u) &\ge \int_0^R r^{N-1}\left[\frac{\mu(r)}{q}|u|^q - \lambda F(r,u)\right] dr \\ &\ge \int_0^R r^{N-1}\left[\frac{\mu(r)}{q} - \frac{\mu_M}{2q}\right]|u|^q dr \ge \frac{\mu_M}{2q}\int_0^R r^{N-1}|u|^q dr. \end{aligned}$$

Let $\rho \in (0, \min\{\delta, \|u_{\lambda,1}\|_\infty/2\})$. We *claim* that

$$\inf_{K_0 \cap \partial B_\rho} \int_0^R r^{N-1}|u|^q dr > 0, \tag{3.2.8}$$

which will imply (3.2.7) with appropriate ρ. To prove (3.2.8), suppose by contradiction that there exists a sequence $\{u_n\} \subset K_0 \cap \partial B_\rho$ with

$$\int_0^R r^{N-1}|u_n|^q dr \to 0 \quad \text{as } n \to \infty.$$

As $\{u_n\}$ is bounded in $W^{1,\infty}$, passing to a subsequence if necessary, we may assume that $\{u_n\}$ is convergent in C to some u. This implies that $\|u\|_\infty = \rho$ and

$$\int_0^R r^{N-1}|u_n|^q dr \to \int_0^R r^{N-1}|u|^q dr \quad \text{as } n \to \infty.$$

It follows that $u = 0$, a contradiction with $\|u\|_\infty = \rho > 0$. So, (3.2.8) holds true, as claimed. Now, by Theorem 1.14, there exists a nontrivial critical point $u_{\lambda,2}$ of I_λ with $I_\lambda(u_{\lambda,2}) > 0$. Clearly, $u_{\lambda,1} \ne u_{\lambda,2}$ and the conclusion follows from Lemma 3.2.3. ■

Example 3.2.5. If $q > 1$ and $m, n > 0$, then there exists $\Lambda > 0$ such that the problem

$$\begin{cases} \mathcal{M}v + \lambda|x|^n\sqrt{\sin v} - |x|^m v^{q-1} = 0 & \text{in} \quad \mathcal{B}(R), \\ \\ v = 0 \quad \text{on} \quad \partial\mathcal{B}(R) \end{cases}$$

has at least one nontrivial radial solution for all $\lambda > \Lambda$.

Example 3.2.6. If $q \in (1, 1.5)$ and $\gamma > 0$, then there exists $\Lambda > 0$ such that the problem

$$\begin{cases} \mathcal{M}v + \lambda\sqrt{v(1-v)} - \gamma v^{q-1} = 0 \quad \text{in} \quad \mathcal{B}(R), \\ \\ v = 0 \quad \text{on} \quad \partial\mathcal{B}(R) \end{cases}$$

has at least two nontrivial radial solutions for all $\lambda > \Lambda$.

Remark 3.2.7. On account of Remark 3.2.2 (ii), it is easy to see that Theorem 3.2.4 still remains true (with the same proof, but with f defined by (3.2.4)) if hypothesis (H_b) is replaced by

(H_b') *the continuous function* $b : [0, R]\times[0, \infty) \to [0, \infty)$ *is such that* $b(r, 0) = 0$ *for all* $r \in [0, R]$ *and* $b(r, s) > 0$ *for all* $(r, s) \in (0, R) \times (0, A)$, *with some* $A > 0$.

Example 3.2.8. If $q > 1$ and $m, n, l > 0$, then there exists $\Lambda > 0$ such that the problem

$$\begin{cases} \mathcal{M}v + \lambda|x|^n v^l - |x|^m v^{q-1} = 0 \quad \text{in} \quad \mathcal{B}(R), \\ \\ v = 0 \quad \text{on} \quad \partial\mathcal{B}(R) \end{cases}$$

has at least one nontrivial radial solution for all $\lambda > \Lambda$.

Example 3.2.9. If $\gamma, l > 0$ and $q \in (1, l + 1)$, then there exists $\Lambda > 0$ such that the problem

$$\begin{cases} \mathcal{M}v + \lambda v^l - \gamma v^{q-1} = 0 \quad \text{in} \quad \mathcal{B}(R), \\ \\ v = 0 \quad \text{on} \quad \partial\mathcal{B}(R) \end{cases}$$

has at least two nontrivial radial solutions for all $\lambda > \Lambda$.

3.3 Existence of solutions for Neumann problems with singular nonlinearities

In this section we are concerned with existence of positive solutions for Neumann problems

$$(r^{N-1}\phi(u'))' + r^{N-1}f(u) = r^{N-1}h(r) \text{ in } [R_1, R_2], \quad u'(R_1) = 0 = u'(R_2), \tag{3.3.1}$$

respectively,

$$(\phi(u'))' - f(u) = h(r) \text{ in } [R_1, R_2], \quad u'(R_1) = 0 = u'(R_2), \tag{3.3.2}$$

where $N \geq 1$ is an integer, $0 \leq R_1 < R_2$, functions $h : [R_1, R_2] \to \mathbb{R}$ and $f : (0, +\infty) \to \mathbb{R}$ are continuous and ϕ is a homeomorphism satisfying hypothesis (H_ϕ) (see Paragraph 3.1.1). Since the class of singular ϕ contains the one given in (3.1.2) as special case, problem (3.5) is nothing else but a problem of type (3.3.1). Also, problem (3.6) with $N = 1$ is of type (3.3.2).

Here and hereafter, we denote by C the Banach space of all continuous functions defined on $[R_1, R_2]$ endowed with the usual norm $\|\cdot\|_\infty$ and by C^1 the Banach space of all continuously differentiable functions on $[R_1, R_2]$ considered with the norm

$$\|u\| = \|u\|_\infty + \|u'\|_\infty \quad (u \in C^1).$$

Also, from now on, for any continuous function $u \in C$, we write

$$u_M = \min_{[R_1,R_2]} u, \quad u^- = \max\{-u, 0\} \quad \text{and} \quad \widetilde{u} = u - \overline{u},$$

where

$$\overline{u} = \frac{N}{R_2^N - R_1^N} \int_{R_1}^{R_2} r^{N-1} u(r) dr.$$

Clearly, one has

$$\int_{R_1}^{R_2} r^{N-1} \widetilde{u}(r) dr = 0. \tag{3.3.3}$$

Using topological arguments and the lower and upper solutions method, A.C. Lazer and S. Solimini investigated in [93] the existence of positive solutions for periodic problems suggested by the model equation $u'' = \rho u^{-\mu} + e(t)$ with $\mu > 0$, $\rho \neq 0$ and e continuous. Their approach also works with minor changes in the Neumann case. Starting with this paper, the interest for the study of periodic singular problems increased. Hence, during the years, this type of results have been generalized or extended for p- or ϕ-Laplacians.

For periodic boundary value problems with singular nonlinearities and p-Laplacian, we refer to the papers of P. Jebelean and J. Mawhin [78], [79]. In [22], C. Bereanu and J. Mawhin prove Lazer-Solimini type results for singular ϕ-Laplacian and periodic boundary conditions, while in [24] they consider the case of $\phi : \mathbb{R} \to (-a, a)$ $(0 < a \leq +\infty)$.

Also, existence results for classical ϕ-Laplacian have been obtained by I. Rachunková and M. Tvrdý in [117], using the method of lower and upper solutions and a continuation theorem due to R. Manásevich and J. Mawhin [98], which also has been applied in [24], [78] and [79]. Note that all the above mentioned papers deal with periodic boundary conditions.

Here we shall employ the ideas from [78] (also see [22], [77], [79]) in order to prove the existence of at least one positive solution for problems (3.3.1), respectively (3.3.2). We note that in the repulsive case, for $N \geq 2$, technical difficulties arise and so, the Neumann problem

$$(r^{N-1}\phi(u'))' - r^{N-1}f(u) = r^{N-1}h(r) \text{ in } [R_1, R_2], \quad u'(R_1) = 0 = u'(R_2)$$

still remains an open one. However, we point out that in the classical case, existence results were obtained by M.A. Del Pino and G.E. Hernandez in [54]. More exactly, they studied the solvability of the Neumann problem

$$-\Delta v + v^{-\mu} = \ell \quad \text{in } B, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial B, \tag{3.3.4}$$

where B is the unit ball in $\mathbb{R}^N$, $N \geq 2$, $\mu > 1$ and ℓ is continuous on $\overline{B}$. Using some a priori estimates for the solutions of a family of approximate problems and variational arguments, they proved the existence of at least one radial solution for problem (3.3.4) provided that $\int_B \ell > 0$.

The rest of the section is organized as follows. The existence of solutions for problem (3.3.1) is studied in the first paragraph, using the method of lower and upper solutions. In the second paragraph, we obtain by degree arguments the existence of at least one positive solution for problem (3.3.2).

3.3.1 Attractive restoring forces

Throughout this paragraph we assume that $R_1 > 0$ and ϕ satisfies hypothesis (H_ϕ). Below, if $u, v \in C$ are such that $u(r) \leq v(r)$ for all $r \in [R_1, R_2]$, we shall write $u \leq v$. Also, $u > 0$ will mean that $u(r) > 0$ for all $r \in [R_1, R_2]$. Of course, by *a solution* of (3.3.1), we mean a function $u \in C^1$ with $\|u'\|_\infty < \eta$, such that $\phi(u') \in C^1$ and (3.3.1) is satisfied.

Definition 3.3.1. A *lower solution* (resp. *upper solution*) of problem (3.3.1) is a function $\alpha \in C^1$, $\alpha > 0$ with $\|\alpha'\|_\infty < \eta$, $\phi(\alpha') \in C^1$, $\alpha'(R_1) \geq 0 \geq \alpha'(R_2)$ (resp. $\beta \in C^1$, $\beta > 0$ with $\|\beta'\|_\infty < \eta$, $\phi(\beta') \in C^1$, $\beta'(R_1) \leq 0 \leq \beta'(R_2)$) and

$$(r^{N-1}\phi(\alpha'(r)))' \geq r^{N-1}(h(r) - f(\alpha(r))) \;\; (r \in [R_1, R_2])$$

$$\big(\text{resp. } (r^{N-1}\phi(\beta'(r)))' \leq r^{N-1}(h(r) - f(\beta(r))) \;\; (r \in [R_1, R_2]\big)\,.$$

Lemma 3.3.2. *If problem* (3.3.1) *has a lower solution α and an upper solution β such that $\alpha \leq \beta$, then* (3.3.1) *has a solution u with $\alpha \leq u \leq \beta$.*

Proof. From [11, Theorem 4.2] we have that problem

$$(r^{N-1}\phi(u'))' = r^{N-1}g(r,u), \quad u'(R_1) = 0 = u'(R_2),$$

with g defined by

$$g(r,x) = h(r) - f(\max\{x, \alpha_M\}) \quad ((r,x) \in [R_1, R_2] \times \mathbb{R})$$

has a solution u, with $\alpha \le u \le \beta$, so u in fact solves problem (3.3.1). ■

Theorem 3.3.3. *If there exists a constant $\alpha > 0$ such that $f(\alpha) \ge \|h\|_\infty$ and f satisfies*

$$\limsup_{u \to +\infty} f(u) < \overline{h}, \tag{3.3.5}$$

then problem (3.3.1) *has at least one positive solution $u \ge \alpha$.*

Proof. Clearly, α is a lower solution of problem (3.3.1). On the other hand, from assumption (3.3.5), there exists a constant $\delta > \alpha$ such that

$$f(u) < \overline{h}, \quad (\forall)\ u \ge \delta. \tag{3.3.6}$$

Let v be the solution of problem

$$(r^{N-1}\phi(v'))' = r^{N-1}\widetilde{h}(r) \text{ in } [R_1, R_2], \quad v'(R_1) = 0 = v'(R_2);$$

this is known to exist from (3.3.3) and [11, Theorem 2.3]. With δ_1 sufficiently large, so that $\delta_1 + v(r) \ge \delta$ for all $r \in [R_1, R_2]$, setting $\beta(r) := \delta_1 + v(r)$ and using (3.3.6), we have

$$\begin{aligned}(r^{N-1}\phi(\beta'(r)))' + r^{N-1}f(\beta(r)) &= (r^{N-1}\phi(v'(r)))' + r^{N-1}f(\delta_1 + v(r)) \\ &= r^{N-1}\widetilde{h}(r) + r^{N-1}f(\delta_1 + v(r)) < r^{N-1}(\widetilde{h}(r) + \overline{h}) = r^{N-1}h(r),\end{aligned}$$

so β is an upper solution of problem (3.3.1), $\beta \ge \alpha$ and the result follows from Lemma 3.3.2. ■

Corollary 3.3.4. *Assume that $f : (0,+\infty) \to (0,+\infty)$ satisfies*

$$f(u) \to +\infty \quad \textit{as} \quad u \to 0+, \tag{3.3.7}$$

$$f(u) \to 0 \quad \textit{as} \quad u \to +\infty. \tag{3.3.8}$$

Then, problem (3.3.1) *has at least one solution if and only if $\overline{h} > 0$.*

Example 3.3.5. For every $\theta, \mu > 0$, the problem

$$\mathcal{M}v + \frac{\theta}{v^\mu} = h(|x|) \text{ in } \mathcal{A}, \quad \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\mathcal{A},$$

has at least one positive solution if and only if $\overline{h} > 0$.

Remark 3.3.6. We emphasize that if $N = 1$, then $R_1 = 0$ is also allowed.

3.3.2 Repulsive restoring forces

In the sequel, using Leray-Schauder degree [94] (see also e.g., [96], [101], [115]), we shall prove that (3.3.2) has at least one positive solution. Here, ϕ will be a homeomorphism satisfying (H_ϕ) – as previously, and $R_1 \geq 0$.

Recall that by *a solution* of problem (3.3.2) we mean a function $u \in C^1$ with $\|u'\|_\infty < \eta$, such that $\phi(u') \in C^1$ and (3.3.2) is satisfied.

Next, we denote by $C^1_\dagger$ the closed subspace of C^1 defined by

$$C^1_\dagger = \left\{u \in C^1 : u'(R_1) = 0 = u'(R_2)\right\}$$

and consider the continuous projectors $P,\ Q : C \to C$ given by

$$Pu = u(R_1), \quad Qu = \overline{u} := \frac{1}{R_2 - R_1}\int_{R_1}^{R_2} u(r)dr.$$

Also, we shall need the linear operator $H : C \to C^1$ defined by

$$Hu(r) = \int_{R_1}^{r} u(t)dt, \quad (r \in [R_1, R_2]).$$

The following fixed point result is proved in [12] (see also [22], [102]).

Proposition 3.3.7. *Let $F : C^1_\dagger \to C$ be a continuous operator which takes bounded sets into bounded sets and consider the abstract Neumann problem*

$$(\phi(u'))' = F(u), \quad u'(R_1) = 0 = u'(R_2). \tag{3.3.9}$$

A function $u \in C^1_\dagger$ is solution of (3.3.9) *if and only if it is a fixed point of the compact operator $M_\dagger : C^1_\dagger \to C^1_\dagger$ defined by*

$$M_\dagger = P + QF + H \circ \phi^{-1} \circ H \circ (I - Q) \circ F.$$

Furthermore, $\|(M_\dagger(u))'\|_\infty < \eta$ for all $u \in C^1_\dagger$.

Finally, we associate to a continuous function $g : [R_1, R_2] \times \mathbb{R} \to \mathbb{R}$ its Nemytskii operator $N_g : C \to C$ (see Section 3.1) and in view of Proposition 3.3.7, taking $F = N_g|_{C^1_\dagger}$, associated to problem

$$(\phi(u'))' = g(r, u), \quad u'(R_1) = 0 = u'(R_2),$$

will be the compact fixed point operator $\mathcal{N} : C^1_\dagger \to C^1_\dagger$ defined by (see also [11, Remark 2.2]):

$$\mathcal{N} = P + QN_g + H \circ \phi^{-1} \circ H \circ (I - Q) \circ N_g. \tag{3.3.10}$$

First, in order to obtain the main result of this paragraph, we shall prove the following two lemmas.

Lemma 3.3.8. *If*

$$\liminf_{u \to 0+} \big(f(u) + \min\{0, \overline{h}\}\big) > 0 \tag{3.3.11}$$

and

$$\limsup_{u \to +\infty} f(u) < -\overline{h} \tag{3.3.12}$$

hold true, then there exist constants $C_1 > C_0 > 0$ such that

(i) f is positive on $(0, C_0]$, and $f(u) + \overline{h} > 0$ (resp. < 0) for all $u \in (0, C_0]$ (resp. $u \geq C_1$);

(ii) for each $\lambda \in [0,1]$ and each possible positive solution u of problem

$$\begin{cases} (\phi(u'))' = (1-\lambda)(Qf(u) + \overline{h}) + \lambda(f(u) + h(r)) \text{ in } [R_1, R_2], \\ u'(R_1) = 0 = u'(R_2), \end{cases} \tag{3.3.13}$$

there exist $r_0, \ r_1 \in [R_1, R_2]$ such that $u(r_0) > C_0$ and $u(r_1) < C_1$.

Proof. Let $\lambda \in [0,1]$ and u be a possible positive solution of (3.3.13). Then, it is easy to see that

$$Qf(u) + \overline{h} = 0. \tag{3.3.14}$$

Hypothesis (3.3.11) implies the existence of a constant $C_0 > 0$ such that $f(u) > 0$ and $f(u) + \overline{h} > 0$, whenever $0 < u \leq C_0$. Consequently, if $0 < u(r) \leq C_0$ for all $r \in [R_1, R_2]$, we have

$$f(u(r)) + \overline{h} > 0, \quad (\forall) \ r \in [R_1, R_2]$$

and hence, by virtue of (3.3.3), one gets

$$\int_{R_1}^{R_2} (f(u(r)) + h(r))dr = \int_{R_1}^{R_2} (f(u(r)) + \overline{h})dr > 0,$$

i.e.,

$$Qf(u) + \overline{h} > 0.$$

This and (3.3.14) imply that $u(r_0) > C_0$ for some $r_0 \in [R_1, R_2]$. On the other hand, from assumption (3.3.12), there exists a constant $C_1 > C_0$ such that $f(u)+\overline{h}<0$ for all $u \geq C_1$. Thus, arguing as above, if $u(r) \geq C_1$ for all $r \in [R_1, R_2]$, one obtains

$$Qf(u)+\overline{h}<0.$$

From this and (3.3.14), we deduce that if u is a solution of (3.3.13), then there exists some $r_1 \in [R_1, R_2]$ such that $u(r_1) < C_1$. ■

Lemma 3.3.9. *Assume* (3.3.11)*,* (3.3.12)*, together with*

$$\liminf_{u\to+\infty} f(u) > -\infty, \tag{3.3.15}$$

$$\int_0^1 f(u)du = +\infty \tag{3.3.16}$$

and let C_0 from Lemma 3.3.8. Then, there exists $\varepsilon \in (0, C_0)$ and $\widetilde{C}_0 > C_1$ such that any positive solution u of (3.3.13) *satisfies $\varepsilon < u(r) < \widetilde{C}_0$ for all $r \in [R_1, R_2]$.*

Proof. For $\lambda \in (0,1]$, on account of (3.3.14), problem (3.3.13) is equivalent to

$$(\phi(u'))' = \lambda(f(u)+h(r)) \text{ in } [R_1, R_2], \quad u'(R_1) = 0 = u'(R_2). \tag{3.3.17}$$

Now, let u be a positive solution of (3.3.17) and let

$$x(r) = \phi(u'(r)), \tag{3.3.18}$$

i.e.,

$$u'(r) = \phi^{-1}(x(r)), \tag{3.3.19}$$

for all $r \in [R_1, R_2]$. From (3.3.18) and (3.3.17), we have

$$x'(r) = \lambda(f(u)+h(r)), \quad (\forall)\ r \in [R_1, R_2]. \tag{3.3.20}$$

Multiplying (3.3.19) by $x'(r)$ and (3.3.20) by $u'(r)$, then subtracting, we get

$$x'(r)\phi^{-1}(x(r)) = \lambda\left(f(u)+h(r)\right)u'(r), \tag{3.3.21}$$

for all $r \in [R_1, R_2]$. Now, (3.3.21) rewrites

$$\left(\int_0^{x(r)} \phi^{-1}(s)ds\right)' - \lambda f(u(r))u'(r) = \lambda h(r)u'(r),$$

which gives

$$\int_0^{x(r)} \phi^{-1}(s)ds - \int_0^{x(r_0)} \phi^{-1}(s)ds - \lambda \int_{u(r_0)}^{u(r)} f(s)ds = \lambda \int_{r_0}^{r} h(s)u'(s)ds,$$

with r_0 given by Lemma 3.3.8 (ii) and using the fact that $\int_0^v \phi^{-1}(s)ds \geq 0$ for all $v \in \mathbb{R}$, we deduce that

$$\lambda \int_{u(r)}^{u(r_0)} f(s)ds \leq \int_0^{x(r_0)} \phi^{-1}(s)ds + \lambda \int_{r_0}^{r} h(s)u'(s)ds, \quad (\forall)\ r \in [R_1, R_2]. \tag{3.3.22}$$

Next, from Lemma 3.3.8 (i) and hypothesis (3.3.15) there is some $\rho > 0$ such that $f(u) \geq -\rho$ for all $u > 0$. Using this, we have

$$\begin{aligned} |f(u) + h(r)| &\leq |f(u) + \rho - \rho| + |h(r)| \\ &\leq |f(u) + \rho| + \rho + h(r) + 2h^-(r) \\ &= f(u) + h(r) + 2(\rho + h^-(r)), \end{aligned} \tag{3.3.23}$$

for all $(r, u) \in [R_1, R_2] \times (0, +\infty)$. From (3.3.17), (3.3.14) and (3.3.23), it follows that

$$\|(\phi(u'))'\|_1 = \lambda \|f(u) + h\|_1 \leq 2\lambda(\rho(R_2 - R_1) + \|h^-\|_1), \tag{3.3.24}$$

where by $\|\cdot\|_1$ we have denoted the usual norm in $L^1(R_1, R_2)$.

From (3.3.24) and the fact that

$$\phi(u'(r)) = \int_{R_1}^{r} (\phi(u'(s)))'\, ds \quad (r \in [R_1, R_2]),$$

we infer

$$|\phi(u'(r))| \leq 2\lambda(\rho(R_2 - R_1) + \|h^-\|_1),$$

i.e., by (3.3.18),

$$|x(r)| \leq \lambda C_2, \quad (r \in [R_1, R_2]), \tag{3.3.25}$$

with

$$C_2 = 2\left(\rho(R_2 - R_1) + \|h^-\|_1\right).$$

Now, using (3.3.22) and (3.3.25), we obtain

$$\begin{aligned} \int_{u(r)}^{u(r_0)} f(s)ds &\leq \frac{1}{\lambda}\left(\Big|\int_0^{x(r_0)} \phi^{-1}(s)ds\Big| + \lambda \int_{R_1}^{R_2} |h(s)u'(s)|ds\right) \\ &\leq \eta(C_2 + \|h\|_1) := C_3. \end{aligned} \tag{3.3.26}$$

On the other hand, with r_1 from Lemma 3.3.8 (ii), one has

$$u(r) = u(r_1) + \int_{r_1}^{r} u'(s)ds < C_1 + \eta(R_2 - R_1) := \widetilde{C}_0, \tag{3.3.27}$$

for all $r \in [R_1, R_2]$. Therefore, if r is such that $u(r) \leq C_0$, from (3.3.26), we get

$$\int_{u(r)}^{C_0} f(s)ds + \int_{C_0}^{u(r_0)} f(s)ds \leq C_3,$$

i.e., by virtue of (3.3.27),

$$\int_{u(r)}^{C_0} f(s)ds \leq C_3 + \int_{C_0}^{\widetilde{C}_0} |f(s)|ds := C_4. \tag{3.3.28}$$

Then, from hypothesis (3.3.16), we can find $\varepsilon \in (0, C_0)$ such that

$$\int_{\varepsilon}^{C_0} f(s)ds > C_4$$

and hence, since f is positive on $(0, C_0]$ (see Lemma 3.3.8 (i)), the inequality (3.3.28) implies that $u(r) > \varepsilon$. So, either $u(r) > C_0$ or $u(r) > \varepsilon$, which means that $u(r) > \varepsilon$ for all $r \in [R_1, R_2]$. If $\lambda = 0$, from (3.3.14), the solutions of (3.3.13) are the constant functions u, which by Lemma 3.3.8 satisfy $C_0 < u < C_1$ and hence $u > \varepsilon$. ∎

Theorem 3.3.10. *Assume that f satisfies* (3.3.11), (3.3.12), (3.3.15) *and* (3.3.16). *Then, problem* (3.3.2) *has at least one positive solution.*

Proof. First, with ε and $\widetilde{C}_0$ from Lemma 3.3.9, we introduce the open bounded set

$$\Omega := \left\{ u \in C^1_\dagger : \varepsilon < u(r) < \widetilde{C}_0 \ (r \in [R_1, R_2]), \ \|u'\|_\infty < \eta \right\}.$$

Next, let $\widetilde{g}$ be defined by

$$\widetilde{g}(x) = \begin{cases} f(\varepsilon), & \text{if } x < \varepsilon; \\ f(x), & \text{if } \varepsilon \leq x \leq \widetilde{C}_0; \\ f(\widetilde{C}_0), & \text{if } x > \widetilde{C}_0 \end{cases}$$

and consider the family of Neumann boundary value problems

$$\begin{cases} (\phi(u'))' = (1 - \lambda)QN_g(u) + \lambda N_g(u) \text{ in } [R_1, R_2], \\ \\ u'(R_1) = 0 = u'(R_2), \end{cases} \tag{3.3.29}$$

where $g(r, x) = h(r) + \widetilde{g}(x)$ $((r, x) \in [R_1, R_2] \times \mathbb{R})$. For each $\lambda \in [0, 1]$, the nonlinear operator $M_\dagger$ on $C^1_\dagger$ associated to (3.3.29) by Proposition 3.3.7 is the operator $\mathcal{H}(\lambda, \cdot)$, where $\mathcal{H}$ is defined on $[0,1] \times C^1_\dagger$ by

$$\mathcal{H}(\lambda, u) = Pu + QN_g(u) + H \circ \phi^{-1} \circ [\lambda H(I-Q)N_g](u).$$

For all $\lambda \in [0,1]$ and $u \in \partial\Omega$, we have that $u \neq \mathcal{H}(\lambda, u)$. Indeed, suppose by contradiction that there exists $\lambda_0 \in [0,1]$ and $u_0 \in \partial\Omega$ such that $u_0 = \mathcal{H}(\lambda_0, u_0)$. Then u_0 is a solution of (3.3.29) with $\lambda = \lambda_0$. Since $u_0 \in \partial\Omega \subset \overline{\Omega}$, one has that $u_0 \in C^1_\dagger$ and $\varepsilon \leq u(r) \leq \widetilde{C}_0$ for all $r \in [R_1, R_2]$. Hence, $g(r, u_0) = h(r) + f(u_0)$ and so u_0 is also solution for problem (3.3.13). But, from Lemma 3.3.9 we know that all possible solutions of (3.3.13) are contained in $\Omega = \overline{\Omega} \backslash \partial\Omega$, a contradiction. So, for all $\lambda \in [0,1]$, if $u = \mathcal{H}(\lambda, u)$, then $u \in \Omega$ and solves (3.3.13).

From the invariance under homotopy of the Leray-Schauder degree – denoted by d_{LS}, one has

$$d_{LS}[I - \mathcal{H}(0, \cdot), \Omega, 0] = d_{LS}[I - \mathcal{H}(1, \cdot), \Omega, 0].$$

On the other hand, clearly $\mathcal{H}(0, \cdot) = P + QN_g$ and $\mathcal{H}(1, \cdot) = \mathcal{N}$ (see (3.3.10)). Using the reduction property of the Leray-Schauder degree we deduce that

$$\begin{aligned} d_{LS}[I - \mathcal{H}(0, \cdot), \Omega, 0] &= d_B[(I - P - QN_g) \mid_{\overline{\Omega} \cap \mathbb{R}}, \Omega \cap \mathbb{R}, 0] \\ &= d_B[-Qf - \overline{h}, (\varepsilon, \widetilde{C}_0), 0] = 1, \end{aligned}$$

where d_B denotes the Brouwer degree. Therefore,

$$d_{LS}[I - \mathcal{N}, \Omega, 0] \neq 0$$

and the existence property of the Leray-Schauder degree implies that $\mathcal{N}$ has at least one fixed point $u \in \Omega$, which is a solution of problem (3.3.2). ■

Corollary 3.3.11. *If $f : (0, +\infty) \to (0, +\infty)$ satisfies conditions* (3.3.7), (3.3.8) *and* (3.3.16)*, then problem* (3.3.2) *has at least one solution if and only if $\overline{h} < 0$.*

Example 3.3.12. For every $\theta > 0$, $\mu \geq 1$ and $h \in C$, the problem

$$\left(\frac{u'}{\sqrt{1-u'^2}}\right)' - \frac{\theta}{u^\mu} = h(r) \text{ in } [R_1, R_2], \quad u'(R_1) = 0 = u'(R_2)$$

has at least one positive solution if and only if $\overline{h} < 0$. The same holds true for the problem

$$(\tan(u'))' - \frac{\theta}{u^\mu} = h(r) \text{ in } [R_1, R_2], \quad u'(R_1) = 0 = u'(R_2).$$

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